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Running head. Manifold-valued martingales

Abstract. We are given a random variable on a Riemannian manifold and we study the set of manifold-valued martingales converging to this variable; more precisely we are interested in uniqueness and existence theorems. We use stochastic calculus tools; for the existence, we restrict ourselves to Wiener probability spaces and we apply the differential calculus which has been developed on them. The relations with some nonlinear partial differential equations are discussed.
0. Introduction

It is well-known that real-valued martingales can be used in the study of some partial differential equations, such as the heat equation and the Dirichlet problem; let us describe briefly how this can be done. Consider a smooth manifold $U$ and a second order differential operator $\mathcal{L}$ on $U$ without term of order zero; if $\phi$ is a real-valued function defined on $U$, the heat equation consists of finding a real-valued function $f$ defined on $(-\infty, 0] \times U$ and satisfying
\[ \frac{\partial f}{\partial t} + \mathcal{L} f = 0, \quad f(0, \cdot) = \phi. \] (0.1)
The probabilistic solution of (0.1) is as follows; let $\Omega$ be the canonical space of continuous paths from $(-\infty, 0]$ into $U$, let $X_t$ be the canonical process and let $\mathbb{P}^{s,x}$ be the law of the Markov process with infinitesimal generator $\mathcal{L}$ and initial value $x$ at time $s$; then a smooth function $f$ is solution of (0.1) if and only if for any $(s, x)$, the process $f(t, X_t)$, $t \geq s$ is a local $\mathbb{P}^{s,x}$-martingale with final value $\phi(X_0)$. Now let $\overline{U}$ be a manifold with boundary $\partial U$ and interior $U$; if $\phi$ is a real-valued function defined on $\partial U$, the Dirichlet problem consists of finding a real-valued function $f$ defined on $\overline{U}$ and satisfying
\[ \mathcal{L} f = 0 \quad \text{on} \quad U, \quad f = \phi \quad \text{on} \quad \partial U. \] (0.2)
This problem has also a probabilistic counterpart; let $\Omega$ be the canonical space of continuous paths from $[0, \infty)$ into $\overline{U}$ stopped when they quit $U$, let $X_t$ be the canonical process, let
\[ \tau = \inf \{ t \geq 0, \quad X_t \in \partial U \}, \] (0.3)
let $\mathbb{P}^x$ be the law of the Markov process with generator $\mathcal{L}$ stopped at $\tau$ with initial value $x$ at time 0, and suppose that $\tau < \infty$ $\mathbb{P}^x$-almost surely for any $x$; then a smooth function $f$ is solution of (0.2) if and only if for any $x$, the process $f(X_t)$ is a local $\mathbb{P}^x$-martingale converging to $\phi(X_\tau)$. Thus both problems can be solved by considering local martingales converging to some fixed random variable.
Let $V$ be a complete Riemannian manifold; the Riemannian metric defines a Hessian operator; if $g$ is a smooth real-valued function defined on $V$, its Hessian at point $x$ will be denoted by $g''(x)$; it is a bilinear form on $T_x V$; if $U$ and $\mathcal{L}$ are as above and if $f$ is a smooth function from $U$ into $V$, one can define a field $\mathcal{L}_V f$ by the following formula; it is a section of the tangent space $TV$ satisfying

$$(\mathcal{L}_V f(x), (g' \circ f)(x)) = \mathcal{L}(g \circ f)(x)$$

(0.4)

for any smooth real-valued function $g$ satisfying $(g'' \circ f)(x) = 0$ (according to the second-order differential geometry [11], $\mathcal{L}_V f$ is the first-order part of the second-order vector field $g \mapsto L(g \circ f)$); $\mathcal{L}_V f$ is called the tension field of $f$ and the equation $\mathcal{L}_V f = 0$ characterizes $\mathcal{L}$-harmonic maps. By replacing $\mathcal{L}$ by $\mathcal{L}_V$, we can consider equations similar to (0.1) and (0.2), but with $\phi$ and $f$ taking their values in $V$; they are nonlinear partial differential equations which can be studied with analytical methods (see [4], [5]). These equations have also equivalent probabilistic problems provided that one uses the notion of $V$-valued martingale; if $M_t$ is a continuous adapted $V$-valued process, we will say that $M_t$ is a martingale (see [2]) if for any smooth real-valued bounded function $g$ on $V$ and for any stopping times $T_1 \leq T_2$ such that $g''(M_t) \geq 0$ on $\{T_1 \leq t < T_2\}$, then $g(M_t), T_1 \leq t \leq T_2$, is a submartingale. As in the real case, we have to study the set of $V$-valued martingales converging almost surely to some fixed random variable; the aim of this work is to study existence and uniqueness of processes in this set; this program was already completed for the circle in [14]; for general manifolds some results are proved in [6], [9]; we will check that they are implied by our results. The uniqueness will rely on a maximum principle. For the existence, we will suppose that the probability space is a Wiener space and will consider two types of terminal values: the variables which live in a small enough convex domain of $V$, and the variables which may be unbounded but are differentiable in the sense of the differential calculus on the Wiener space.

We will first give some preliminary results concerning the geodesic distance and the differential calculus on the Wiener space. In §2, we will study the uniqueness problem; we will describe classes of martingales in which two martingales converging to the same value are indistinguishable; the basic tool is a maximum principle which will be proved by
means of Itô’s formula. In §3 we will prove an existence theorem when the final value is differentiable in the sense of the differential calculus on the Wiener space; in the proof we will need a stochastic implicit function theorem. In §4 we will obtain a general existence theorem for martingales taking values in a domain of $V$ satisfying some assumptions; regular geodesic balls (see [9]) are a particular case of such domains; in particular, when the sectional curvatures are non-positive, we will obtain the existence for any integrable final value. Finally, in §5, from these results we will deduce probabilistic solutions to the nonlinear heat equation and Dirichlet problem.

Throughout this work, we will assume that $V$ is a complete Riemannian manifold (without boundary) of dimension $d$; in each tangent space $T_xV$, the Riemannian norm and product will be denoted by $|.|$ and $(.,.)$. We will always assume that the injectivity radius $R$ of $V$ is positive and that its sectional curvatures are bounded above; we will let $K$ be the smallest non-negative number dominating all the sectional curvatures. If $f$ is a smooth real function defined on $V$, its derivative $f'(x)$ is a linear form on $T_xV$ and its value on a tangent vector $u$ will be denoted by $f'(x)(u)$; similarly, the Hessian $f''(x)$ is a bilinear form the value of which is denoted by $f''(x)(u,\bar{u})$; if $f$ is defined on $V \times V$ endowed with the product Riemannian metric, if $x = (x_0, x_1)$ is a point, $u = (u_0, u_1)$ a tangent vector, we will consider the partial derivatives

$$f''_{00}(x)(u_0, u_0) = f''(x)((u_0, 0), (u_0, 0)),$$

(0.5)

$$f''_{01}(x)(u_0, u_1) = f''(x)((u_0, 0), (0, u_1))$$

(0.6)

and $f''_{11}$ defined similarly. We will always assume that we are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions; for some results, we will suppose that $\Omega$ is a Wiener space. The quadratic variation of a $V$-valued martingale $M_t$ (computed for the Riemannian metric) will be denoted by $\langle \langle M \rangle \rangle_t$. We will say that a sequence of processes $X^k_t$ converges uniformly in probability to a process $X_t$ if $\sup_t |X^k_t - X_t|$ converges in probability to 0. Different constant numbers will often be denoted by the same letter $C$. Geometric results which are used in this work can be found in [1], [10], whereas the theory of real-valued martingales and the stochastic differential geometry are respectively dealt with in [3] and [6] (among other books and articles).
1. Preliminary results

We will first prove some useful geometrical estimates for the distance function on \( V \times V \). Then we will briefly describe the differential calculus on Wiener spaces.

§1.1 Estimates on the derivatives of the distance

If \( x_0 \) and \( x_1 \) are two points of \( V \), denote by \( \delta(x_0, x_1) \) the geodesic distance between them; the function \( \delta \) is defined on \( V \times V \) and is smooth except on the cut locus and the diagonal \( \{x_0 = x_1\} \). We want to estimate its first and second derivatives when \( V \times V \) is endowed with the product Riemannian metric. Let \( x = (x_0, x_1) \) be a point which is not in the cut locus or the diagonal; there exists a unique minimizing geodesic \( c(t), 0 \leq t \leq 1 \) from \( x_0 \) to \( x_1 \); if \( u_t \) is a vector of \( T_c(t) \), we can decompose \( u_t \) as \( v_t + w_t \) where \( v_t \) is the orthogonal projection of \( u_t \) on \( c'(t) \); the vectors \( v_t \) and \( w_t \) will be respectively called the parallel and orthogonal components of \( u_t \). If \( u = (u_0, u_1) \) is a vector of \( T_x(V \times V) \), \( (v_0, v_1) \) and \( (w_0, w_1) \) will also be called its parallel and orthogonal components.

Lemma 1.1.1. Let \( x \) be a point of \( V \times V \) which is not in the cut locus and such that \( 0 < \delta(x) < \pi/\sqrt{K} \). Let \( u \) be a vector of \( T_x(V \times V) \) and let \( v \) and \( w \) be its parallel and orthogonal components. Then

\[
|\delta'(x)(u)| \leq \sqrt{2}|v|
\]

and

\[
\delta''(x)(u,u) \geq -\sqrt{K} \tan \left( \frac{\sqrt{K} \delta(x)}{2} \right) |w|^2.
\]

Proof. Let \( g(s), s \geq 0 \) be the geodesic line on \( V \times V \) satisfying \( g(0) = x \) and \( g'(0) = u \); then the quantities that we have to estimate are

\[
\delta'(x)(u) = \frac{d}{ds} \delta(g(s)) \bigg|_{s=0}, \quad \delta''(x)(u,u) = \frac{d^2}{ds^2} \delta(g(s)) \bigg|_{s=0}.
\]

For \( s \) small enough, let \( c(s,t), 0 \leq t \leq 1 \) be the unique minimizing geodesic joining the two components of \( g(s) \); put \( c(t) = c(0,t) \). Let \( J(t) \) be the derivative of \( c(s,t) \) with respect to \( s \) at \( s = 0 \); it is the Jacobi field on \( c(t) \) satisfying \( J(0) = u_0 \) and \( J(1) = u_1 \); let \( J'(t) \) be the covariant derivative of \( J \) with respect to \( t \). We also define \( K(t) \) to be the sectional
curvature at the point \( c(t) \) for the plane generated by \( c'(t) \) and \( J(t) \) (zero if these two vectors do not generate a plane); if \( K_0 \) is the curvature tensor of \( V \), then

\[
\left( K_0(J(t), c'(t))c'(t), J(t) \right) = K(t) \left[ \left| J(t) \right|^2 \left| c'(t) \right|^2 - \left( J(t), c'(t) \right)^2 \right].
\]

(1.1.4)

By noticing that \( \delta(g(s)) \) is the norm of \( \frac{\partial}{\partial t} c(s, t) \), one can compute the expressions of (1.1.3) (see the proof of Proposition 3.8.1 of [10]) and obtain

\[
\delta'(x)\langle u \rangle = \frac{(c'(t), J'(t))}{\left| c'(t) \right|}
\]

(1.1.5)

for any \( 0 \leq t \leq 1 \) and

\[
\delta''(x)\langle u, u \rangle = \int_0^1 \frac{\left| J'(t) \right|^2 \left| c'(t) \right|^2 - (J'(t), c'(t))^2}{\left| c'(t) \right|^3} dt - \int_0^1 \left( K_0(J(t), c'(t))c'(t), J(t) \right) \frac{dt}{\left| c'(t) \right|}
\]

\[
= \int_0^1 \frac{\left| J'(t) \right|^2 \left| c'(t) \right|^2 - (J'(t), c'(t))^2}{\left| c'(t) \right|^3} dt
\]

\[
- \int_0^1 K(t) \frac{\left| J(t) \right|^2 \left| c'(t) \right|^2 - (J(t), c'(t))^2}{\left| c'(t) \right|^3} dt.
\]

(1.1.6)

Now the decomposition \( u = v + w \) induces a decomposition \( J = J_v + J_w \) into two Jacobi fields which are its parallel and orthogonal components. Thus it follows from (1.1.5) that \( \delta'(x)\langle u \rangle \) depends only on the component \( v \); moreover one immediately deduces from the triangle inequality that \( \delta \) is \( \sqrt{2} \)-Lipschitz so (1.1.1) is checked. Similarly, \( \delta''(x)\langle u, u \rangle \) depends only on the component \( w \); moreover if \( K = 0 \), the right-hand side of (1.1.6) is non-negative so we only have to consider the case \( K > 0 \). Formula (1.1.6) can be written as

\[
\delta''(x)\langle u, u \rangle = \frac{1}{\delta(x)} \int_0^1 \left| J'_w(t) \right|^2 dt - \delta(x) \int_0^1 K(t) \left| J_w(t) \right|^2 dt.
\]

(1.1.7)

On the other hand, we can deduce from the Jacobi equation

\[
J''(t) + K_0(J(t), c'(t))c'(t) = 0
\]

(1.1.8)

that

\[
(J(t), J''(t)) + K(t) \left[ \left| J(t) \right|^2 \left| c'(t) \right|^2 - (J(t), c'(t))^2 \right] = 0
\]

(1.1.9)
so for \( J = J_w \), this yields
\[
(J_w(t), J''_w(t)) + K(t)\delta^2(x)|J_w(t)|^2 = 0. \tag{1.1.10}
\]
We deduce from (1.1.7) and (1.1.10) that
\[
\delta''(x)\langle u,u \rangle = \frac{1}{2\delta(x)} \int_0^1 \frac{d^2}{dt^2} |J_w(t)|^2 \, dt \tag{1.1.11}
\]
and that
\[
\frac{d^2}{dt^2} |J_w(t)| = \frac{|J_w(t)|^2 |J'_w(t)|^2 - (J_w(t), J''_w(t))^2}{|J_w(t)|^3} - K(t)\delta^2(x)|J_w(t)| \geq -K\delta^2(x)|J_w(t)| \tag{1.1.12}
\]
at points \( t \) such that \( J_w(t) \neq 0 \). Consider the function
\[
j(t) = \alpha \sin(\sqrt{K}\delta(x)t + \beta) \tag{1.1.13}
\]
where \( \alpha > 0 \) and \( 0 \leq \beta \leq \pi - \sqrt{K}\delta(x) \) are defined by
\[
\alpha \sin \beta = |w_0|, \quad \alpha \sin(\sqrt{K}\delta(x) + \beta) = |w_1|. \tag{1.1.14}
\]
In particular
\[
\alpha^2 = \frac{|w|^2}{\sin^2 \beta + \sin^2(\sqrt{K}\delta(x) + \beta)}. \tag{1.1.15}
\]
By comparing (1.1.12) with the corresponding differential equation with boundary conditions, one deduces since \( \sqrt{K}\delta(x) < \pi \) that
\[
\left. \frac{d}{dt} |J_w(t)| \right|_{t=0} \leq j'(0), \quad \left. \frac{d}{dt} |J_w(t)| \right|_{t=1} \geq j'(1). \tag{1.1.16}
\]
Moreover, \( |J_w(0)| \) and \( |J_w(1)| \) are respectively \( j(0) \) and \( j(1) \). Thus from (1.1.11),
\[
\delta''(x)\langle u,u \rangle \geq \frac{1}{\delta(x)} (j(1)j'(1) - j(0)j'(0))
\]
\[
= |w|^2 \sqrt{K} \left( \sin(\sqrt{K}\delta(x) + \beta) \cos(\sqrt{K}\delta(x) + \beta) - \sin \beta \cos \beta \right) \cdot \frac{1}{\sin^2 \beta + \sin^2(\sqrt{K}\delta(x) + \beta)}. \tag{1.1.17}
\]
The derivative with respect to \( \beta \) of the above ratio is
\[
\cos(2\sqrt{K}\delta(x) + 2\beta) - \cos(2\beta) \left[ \sin^2 \beta + \sin^2(\sqrt{K}\delta(x) + \beta) \right]^2
\]
so the minimum with respect to \( \beta \) of the right-hand side of (1.1.17) is obtained for \( \beta = (\pi - \sqrt{K}\delta(x))/2 \) and is exactly the right-hand side of (1.1.2). \( \square \)
\section{Estimates on the partial derivatives of the distance}

\textbf{Lemma 1.2.1.} Let $x$ be a point of $V \times V$ which is not in the cut locus and such that $0 < \delta(x) < \pi/\sqrt{K}$. Let $(u_0, u_1)$ be a tangent vector and denote by $(v_0, v_1)$ and $(w_0, w_1)$ its parallel and orthogonal components. Then

\begin{equation}
|\delta'_0(x)\langle u_0 \rangle| = |v_0|,
\end{equation}

\begin{equation}
\delta''_0(x)\langle u_0, u_0 \rangle \geq \sqrt{K} \cot(\sqrt{K} \delta(x)) |w_0|^2
\end{equation}

and

\begin{equation}
|\delta''_{01}(x)\langle u_0, u_1 \rangle| \leq \frac{\sqrt{K}}{\sin(\sqrt{K} \delta(x))} |w_0| |w_1|
\end{equation}

where the coefficients in (1.2.2) and (1.2.3) are defined to be $1/\delta(x)$ if $K = 0$.

\textbf{Proof.} We will use the notations defined in the proof of Lemma 1.1.1. It is sufficient to prove the lemma for $K > 0$; the case $K = 0$ is obtained by applying the result for $K > 0$ and letting $K \downarrow 0$. Note also that the linear form $\delta'_0(x)$ is the scalar product by $c'(0)$ so we immediately deduce (1.2.1). Put $u = (u_0, 0)$ and $\overline{u} = (0, u_1)$. We can apply (1.1.17) to our vector $u$ with $j(0) = |w_0|$ and $j(1) = 0$ so that

\begin{equation}
\beta = \pi - \sqrt{K} \delta(x), \quad \alpha = |w_0|/\sin \beta.
\end{equation}

We deduce (1.2.2). Let us now prove (1.2.3). Use the decomposition $u = v + w$ and $\overline{u} = \overline{v} + \overline{w}$; by polarizing (1.1.11) we obtain

\begin{equation}
\delta''(x)\langle u, \overline{u} \rangle = \frac{1}{2\delta(x)} \int_0^1 \frac{d^2}{dt^2} \langle J_w(t), J_{\overline{w}}(t) \rangle dt
\end{equation}

\begin{equation}
= \frac{1}{2\delta(x)} \left[ \frac{d}{dt} \langle J_w(t), J_{\overline{w}}(t) \rangle \right]_{t=1} - \frac{d}{dt} \langle J_w(t), J_{\overline{w}}(t) \rangle \right]_{t=0}.
\end{equation}

\begin{equation}
\left| \langle J_w(t), J_{\overline{w}}(t) \rangle \right| \leq |J_w(t)||J_{\overline{w}}(t)|
\end{equation}

and these terms are 0 for $t = 0$ or 1, so

\begin{equation}
\left| \frac{d}{dt} \langle J_w(t), J_{\overline{w}}(t) \rangle \right| \leq \left| \frac{d}{dt} \left( |J_w(t)||J_{\overline{w}}(t)| \right) \right|
\end{equation}

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for \( t = 0 \) or \( 1 \). Thus
\[
\left| \delta''(x)\langle u, \pi \rangle \right| \leq \frac{1}{2\delta(x)} \left[ |w| \left. \frac{d}{dt} \left| J\pi(t) \right| \right|_{t=0} - |\pi| \left. \frac{d}{dt} \left| J\pi(t) \right| \right|_{t=1} \right]. \tag{1.2.8}
\]
We deduce from (1.1.16) that we can estimate the derivative of \( |J\pi(t)| \) at \( t = 1 \) by \( j'(1) \) where \( j(t) \) is given by (1.1.13) and (1.2.4); thus
\[
- \left. \frac{d}{dt} \left| J\pi(t) \right| \right|_{t=1} \leq |w| \frac{\sqrt{K}\delta(x)}{\sin(\sqrt{K}\delta(x))}. \tag{1.2.9}
\]
The derivative of \( |J\pi(t)| \) is estimated in a similar way so that we check
\[
\left| \delta''(x)\langle u, \pi \rangle \right| \leq |w| |\pi| \frac{\sqrt{K}}{\sin(\sqrt{K}\delta(x))}. \tag{1.2.10}
\]
This is exactly (1.2.3).

\[\Box\]

§1.3 The differential calculus on Wiener spaces

In this subsection, we suppose that the probability space is a Wiener space, we review briefly the differential calculus which can be developed on it (see for instance [12]) and apply it to \( V \)-valued variables. So let \( \Omega \) be the space of continuous functions from \([0, \infty) \) into some Euclidean space \( \mathbb{R}^m \), let \( \mathbb{P} \) be the standard Wiener measure and \( W_t \) be the canonical process: it is a \( m \)-dimensional standard Wiener process. Then let \( \mathcal{S}(\mathbb{R}) \) be the set of real-valued smooth functionals \( X = f(W_t_1, \ldots, W_t_k) \) which are bounded as well as all their derivatives. For such a smooth functional, define
\[
D_t X = \sum_{i=1}^k \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_k})1_{\{t \leq t_i\}} \tag{1.3.1}
\]
considered as a \( m \)-dimensional row vector and put
\[
\|X\|^2_D = \mathbb{E}X^2 + \mathbb{E} \int_0^\infty \|D_t X\|^2 dt. \tag{1.3.2}
\]
One can prove that \( \mathcal{S}(\mathbb{R}) \) is closable for this norm and we denote by \( \mathcal{D}(\mathbb{R}) \) the resulting completion; then \( \mathcal{D}(\mathbb{R}) \) is a Hilbert space and if \( X \) is in it, the process \( D_t X \) is well defined almost everywhere. If \( X \) is in \( \mathcal{D}(\mathbb{R}) \) and if \( \phi \) is uniformly Lipschitz, then \( \phi(X) \) is in
\(\mathcal{D}(\mathbb{R})\) (this can be proved by approximating \(\phi\) by smooth functions); if moreover \(\phi\) is smooth, then the derivative of \(\phi(X)\) is given by the classical chain rule; more generally, if \(X_i\) are \(p\) variables in \(\mathcal{D}(\mathbb{R})\) and if \(\phi\) is Lipschitz on \(\mathbb{R}^p\) endowed with the sup norm, then \(\phi(X_1, \ldots, X_p)\) is in \(\mathcal{D}(\mathbb{R})\) and

\[
|D\phi(X_1, \ldots, X_p)| \leq \|\phi\|_{\text{lip}} \sup_i |DX_i|.
\]  

(1.3.3)

A fundamental property of variables of \(\mathcal{D}(\mathbb{R})\) is the representation formula

\[
X = \mathbb{E}X + \sum_{j=1}^{m} \int_0^\infty \mathbb{E}[D_1^j X | \mathcal{F}_t] dW_t^j.
\]  

(1.3.4)

We will need two other properties of \(\mathcal{D}(\mathbb{R})\).

**Lemma 1.3.1.** Let \(X\) be a variable of \(\mathcal{D}(\mathbb{R})\) which takes its values in a discrete subset of \(\mathbb{R}\); then \(X\) is almost surely constant.

*Proof.* Denote the discrete subset by \(A\); there exists a smooth bounded function \(\phi\) with bounded derivative such that \(\phi' = 0\) on \(A\) and \(\phi\) is injective. Thus \(\phi(X)\) is in \(\mathcal{D}(\mathbb{R})\) and \(\phi'(X) = 0\) implies \(D\phi(X) = 0\); thus, from (1.3.4), \(\phi(X)\) is almost surely constant and we can conclude. \(\square\)

**Lemma 1.3.2.** Let \(X^n\) be a sequence of variables of \(\mathcal{D}(\mathbb{R})\) converging in probability to \(X\); suppose that \(DX^n\) is bounded in \(L^2([0, \infty] \times \mathbb{P})\). Then \(X\) is in \(\mathcal{D}(\mathbb{R})\).

*Proof.* Since \(\mathcal{D}(\mathbb{R})\) is a Hilbert space, it is sufficient to check that \(X^n\) is bounded for the norm (1.3.2), and therefore that it is bounded in \(L^2\); but we deduce from (1.3.4) that \(X^n - \mathbb{E}X^n\) is bounded in \(L^2\) and if \(\mathbb{E}X^n\) is unbounded, \(X^n\) cannot converge in probability. \(\square\)

Then let \(\mathcal{D}(V)\) be the set of \(V\)-valued variables \(X\) such that for any function \(\phi : V \to \mathbb{R}\) which is smooth and uniformly Lipschitz, the variable \(\phi(X)\) is in \(\mathcal{D}(\mathbb{R})\); by using the Nash embedding theorem, the manifold \(V\) can be considered as a submanifold of a Euclidean space \(\mathbb{R}^r\), and saying that \(X\) is in \(\mathcal{D}(V)\) is equivalent to saying that each component of \(X\) in
Let $\mathbb{R}^r$ be in $\mathcal{D}(\mathbb{R})$. Moreover if $X$ is in $\mathcal{D}(V)$, we can consider a process $(X, D_1^X, \ldots, D_r^X)$ defined $\mathbb{P} \otimes dt$ almost everywhere, such that $D_j^X \in T_X V$ and

$$D_j^X \phi(X) = (\phi'(X), D_j^X) \quad \mathbb{P} \otimes dt \quad \text{a.e.} \quad (1.3.5)$$

If $X_t$ is a measurable process, we will say that $X$ is in $\mathcal{D}^s(V)$ if $X$ is the uniform limit in probability of a sequence of $V$-valued step processes $Y^n$ (associated to deterministic subdivisions) satisfying

$$\limsup_{n \to \infty} \mathbb{E} \left[ \sum_{j=1}^m \int_0^\infty \sup_t |D_j^{Y^n}_t| \, ds \right] < \infty. \quad (1.3.6)$$

This implies in particular that for any $t$, $X_t$ is in $\mathcal{D}(V)$.

2. The uniqueness property

Fix some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose that $M_t$ and $\overline{M}_t$ are two martingales on the Riemannian manifold $V$ converging almost surely to the same variable as $t \to \infty$; we want to find conditions ensuring that $M = \overline{M}$ almost surely. If $V = \mathbb{R}$, the process $M_t - \overline{M}_t$ is a local martingale converging to 0; it is well known that it is not necessarily zero; however if $\mathcal{T}$ denotes the set of $\mathcal{F}_t$ stopping times and if $M_t$ and $\overline{M}_t$ are in the class of processes $X_t$ such that $(X_{\tau}, \tau \in \mathcal{T})$ is uniformly integrable, then $M_t$ and $\overline{M}_t$ are uniformly integrable martingales, so they coincide. More generally if $V$ is a Cartan-Hadamard manifold (a simply connected manifold with non-positive sectional curvatures) then the geodesic distance $\delta$ is convex on $V \times V$ so $\delta(M_t, \overline{M}_t)$ is a local submartingale; thus if $O$ is some fixed point of $V$ and $M_t$ and $\overline{M}_t$ are in the class of processes $X_t$ such that $(\delta(O, X_{\tau}), \tau \in \mathcal{T})$ is uniformly integrable, we can again conclude that $M$ and $\overline{M}$ coincide. On the other hand, this property does not hold for more general manifolds: consider for instance the circle ([14]). The aim of this section is to describe classes of martingales satisfying this property; the basic tool will be a maximum principle.

§2.1 A maximum principle

In this subsection, we prove a maximum principle and give simple applications of it. More precisely, the maximum principle says that if a subset of $V$ satisfies a convexity
property and if $M_t$ is a $V$-valued martingale converging to a variable in the subset, then
the whole martingale lives in the subset; actually this will generally not be valid for all
martingales but only for those in some class which we now define.

**Definition 2.1.1.** If $\lambda$ is a non-negative measurable function defined on $V$, let $\mathcal{E}_\lambda$ be the
set of $V$-valued martingales $M_t$ satisfying
\[
\mathbb{E} \exp \int_0^\infty \lambda(M_t)d\langle M \rangle_t < \infty. \tag{2.1.1}
\]

First note that if $\lambda$ is locally bounded and if $M_t$ is a martingale which converges
almost surely, then it follows from [16] that $\int \lambda(M_t)d\langle M \rangle_t$ is almost surely finite. On the
other hand, we now verify by a simple application of Itô’s formula that the class $\mathcal{E}_\lambda$ may
contain all $\Delta$-valued martingales, for some small enough domains $\Delta$.

**Proposition 2.1.2.** Let $\Delta$ be an open subset of $V$; suppose that we are given on $\Delta$ a
non-negative locally bounded function $\lambda$ and a $C^2$ function $f$ satisfying $c \leq f(x) \leq C$ for
some positive $c$ and $C$; suppose moreover that $f'' + 2\lambda f \leq 0$ on $\Delta$: this means that
\[
f''(x)\langle u, u \rangle + 2\lambda(x)f(x)|u|^2 \leq 0. \tag{2.1.2}
\]
Then any $\Delta$-valued martingale belongs to $\mathcal{E}_\lambda$.

**Proof.** Define
\[
S_t = f(X_t)\exp \int_0^t \lambda(M_s)d\langle M \rangle_s. \tag{2.1.3}
\]
It follows immediately from Itô’s formula and condition (2.1.2) that $S_t$ is a local super-
martingale; since it is non-negative, we deduce that $\mathbb{E}S_\infty \leq \mathbb{E}S_0$. By using the lower and
upper bounds on $f$, we deduce (2.1.1).

We now state and prove the maximum principle for martingales.

**Theorem 2.1.3.** Let $f$ be a real-valued continuous function defined on $V$ and consider
the subsets $\Delta = \{f < 0\}$ and $F = \{f \leq 0\}$.

(a) Suppose that $f$ is $C^2$ and bounded on $V \setminus F$ and that there exists a non-negative
function \( \lambda \) defined on \( V \setminus F \) and such that \( f'' + 2\lambda f \geq 0 \). Extend the function \( \lambda \) to \( V \) by putting \( \lambda = 0 \) on \( F \). Then any martingale of \( \mathcal{E}_\lambda \) converging to a variable in \( F \) lives in \( F \).

(b) Suppose that \( f \) is \( C^2 \) on \( F \) and that there exists a non-positive locally bounded function defined on \( F \) and such that \( f'' + 2\lambda f \geq 0 \). Then any martingale living in \( F \) and converging to a variable in \( \Delta \) lives in \( \Delta \).

Proof. Let \( M_t \) be a martingale of \( \mathcal{E}_\lambda \) converging to \( M_\infty \in F \). Fix some time \( t_0 \geq 0 \) and put

\[
\tau = \inf \{ t \geq t_0 ; M_t \in F \}.
\]  

The stopping time \( \tau \) may be infinite but in any case, \( M_\tau \in F \). Define the process \( S_t \) by (2.1.3). On the event \( \{ M_{t_0} \notin F \} \), the process \( (S_t, t_0 \leq t < \tau) \) is a non-negative local submartingale with positive initial value and zero limit; since \( f \) is bounded and \( M_t \) is in \( \mathcal{E}_\lambda \), it is actually a uniformly integrable submartingale; we obtain a contradiction so the probability of the event \( \{ M_{t_0} \notin F \} \) is zero. Since \( t_0 \) is arbitrary and \( F \) is closed, we deduce (a). Under the assumptions of (b), let \( M_t \) be a martingale living in \( F \); if \( S_t \) is again defined by (2.1.3), it is a non-positive local submartingale so the event \( \{ \sup_t S_t = 0 \} \) is almost surely equal to \( \{ S_\infty = 0 \} \). Now suppose that \( M_t \) converges to \( M_\infty \in \Delta \); since \( \lambda \) is locally bounded, it follows from [16] that the integral in the exponential of (2.1.3) converges as \( t \to \infty \), so \( S_\infty < 0 \); thus \( \sup_t S_t < 0 \) and (b) is proved.

Remark. We have assumed that \( f \) is \( C^2 \) in order to apply Itô’s formula but we can also use less regular functions by an approximation technique. For instance, in case (a), if \( f \) is the uniform limit of a uniformly bounded sequence of \( C^2 \) functions \( f_n \) and if \( f_n'' \) is uniformly lower bounded, the theorem still holds with \( f'' \) replaced by \( \lim \inf f_n'' \) (apply Fatou’s lemma).

Before the main applications, let us give two consequences of the maximum principle which will not be used subsequently. The first one can be compared with Theorem 3.1 of [9].

**Corollary 2.1.4.** Consider a constant function \( \lambda > 0 \) such that \( \lambda \geq \frac{K^2}{r} \vee \frac{1}{r} \left( \frac{c}{r} \right)^2 \). Then any martingale of \( \mathcal{E}_\lambda \) converging almost surely to a deterministic value is constant.
Remark. In a Cartan-Hadamard manifold, the property holds for martingales of $\bigcup_{\lambda > 0} E_\lambda$; actually in this case, by using the convexity of the distance function, one can prove that if the sectional curvatures are bounded below, it is sufficient to assume that $\langle M \rangle_\infty$ is integrable.

Proof. Let us consider martingales converging to a point $x_0$; define the function

$$f(x) = \sin\left(\sqrt{2\lambda}\delta(x_0, x)\right) \quad \text{if} \quad \delta(x_0, x) \leq \frac{\pi}{2\sqrt{2\lambda}},$$

1 otherwise. Then $f$ is $C^1$ on $V \setminus \{x_0\}$, is $C^2$ except at $x_0$ and on the geodesic sphere $\{\delta(x_0, x) = \pi/\sqrt{8\lambda}\}$. Let us estimate $f''(x)$ for $x$ in the geodesic ball and different from $x_0$; from the estimates of Lemma 1.2.1, if $u$ is a tangent vector with parallel and orthogonal components $v$ and $w$,

$$f''(x)(u, u) = \sqrt{2\lambda}\cos\left(\sqrt{2\lambda}\delta(x_0, x)\right)\delta''_{11}(x_0, x)(u, u)$$

$$- 2\lambda\sin\left(\sqrt{2\lambda}\delta(x_0, x)\right)\left|\delta'_1(x_0, x)(u)\right|^2$$

$$\geq \sqrt{2\lambda K}\sin\left(\sqrt{K}\delta(x_0, x)\right)\left(\cos\left(\sqrt{K}\delta(x_0, x)\right) \wedge 0\right)|w|^2$$

$$- 2\lambda f(x)|v|^2$$

(2.1.6)

because $\sqrt{2\lambda} \geq \sqrt{K}/2$. But if $0 < z < \pi/2$, one has

$$\frac{\cos(2z) \cos z}{\sin(2z)} = \frac{1 - 2\sin^2 z}{2\sin z} \geq \frac{1}{2} \sin z$$

(2.1.7)

so

$$f''(x)(u, u) \geq - \frac{\lambda K}{2} \sin\left(\sqrt{K}\delta(x_0, x)\right)|w|^2 - 2\lambda f(x)|v|^2$$

$$\geq - 2\lambda f(x)|v|^2.$$  

(2.1.8)

By regularizing $f$, we can apply Theorem 2.1.3 and deduce the corollary. □

The maximum principle can also be used for convex domains of $V$. Let $\Delta$ be an open subset of $V$ and suppose that its boundary $\partial \Delta$ is a smooth submanifold of dimension $d - 1$; consider the function

$$f_0(x) = -( -1)^{d(x)} \delta(x, \partial \Delta).$$

(2.1.9)
Each point of ∂Δ admits on a neighbourhood a system of local coordinates \((y_1, \ldots, y_d)\) such that \(f_0(x) = y_d\); in particular \(f_0\) is smooth on a neighbourhood of \(\partial \Delta\). We will say that the boundary is convex if \(f''_0 \geq 0\) on \(\partial \Delta\) (\(f''_0\) is the second fundamental form of \(\partial \Delta\) relative to the normal vector field exiting \(\Delta\)).

**Corollary 2.1.5.** Let \(\Delta\) be a relatively compact open subset of \(V\) with smooth convex boundary. There exists a positive constant \(\lambda\) such that any martingale of \(E_\lambda\) converging to a variable in \(\Delta\) lives in \(\Delta\).

**Proof.** We can construct a smooth function \(f\) on \(V\) such that \(f = f_0\) on a neighbourhood of \(\partial \Delta\), \(f > 0\) on \(V \setminus \Delta\), \(f < 0\) on \(\Delta\) and \(f = 1\) except on a compact subset. The domain \(\Delta\) is convex so \(f''(x)\) is non-negative on \(\{f(x) = 0\}\). Since \(f''\) is Lipschitz, we can deduce that \(f''/|f|\) is lower bounded by some constant number; we conclude with Theorem 2.1.3.

\[\square\]

### §2.2 The uniqueness results

We can easily deduce from the maximum principle the following uniqueness result.

**Theorem 2.2.1.** Consider a constant function \(\lambda > 0\) such that \(\lambda \geq \frac{K}{2} + \frac{1}{2}(\frac{\pi}{\rho})^2\). Then for any measurable variable, there is at most one martingale of the class \(E_\lambda\) converging almost surely to this variable.

**Proof.** Put \(\rho = (2\lambda)^{-1/2}\pi\) and consider the function

\[f(x) = \sin \left(\frac{\pi \delta(x)}{2\rho}\right)1_{\{\delta(x) < \rho\}} + 1_{\{\delta(x) \geq \rho\}}\]  \hspace{1cm} (2.2.1)

defined on \(V \times V\). Then on \(\{0 < \delta < \rho\}\), the function \(f\) is smooth and

\[f''(x)\langle u, u \rangle = \frac{\pi}{2\rho} \cos \left(\frac{\pi \delta(x)}{2\rho}\right) \delta''(x)\langle u, u \rangle\]  
\[= \left(\frac{\pi}{2\rho}\right)^2 f(x)\delta'(x)\langle u, u \rangle^2.\]  \hspace{1cm} (2.2.2)

\(\delta\)From Lemma 1.1.1 we deduce

\[f''(x)\langle u, u \rangle \geq -\frac{\pi}{2\rho} \sqrt{K} \tan \left(\frac{1}{2}\sqrt{K}\delta(x)\right) \cos \left(\frac{\pi \delta(x)}{2\rho}\right)|w|^2 - \frac{1}{2}\left(\frac{\pi}{\rho}\right)^2 f(x)|v|^2\]

\[\geq -\frac{1}{2}\left(\frac{\pi}{\rho}\right)^2 f(x)|w|^2 - \frac{1}{2}\left(\frac{\pi}{\rho}\right)^2 f(x)|v|^2\]  \hspace{1cm} (2.2.3)
where the last inequality follows from $K \leq \left( \frac{\pi}{2} \right)^2$. Since $|u|^2 = |v|^2 + |w|^2$ and from the definition of $\rho$, we deduce that $f'' + \lambda f \geq 0$; moreover the set $\{ f = 0 \}$ is exactly the diagonal of $V \times V$. On the other hand, if $M_t$ and $M_t'$ are two martingales of $\mathcal{E}_\lambda$ such that $M_\infty = M'_\infty$, then $(M_t, M'_t)$ is a $V \times V$-valued martingale converging to a variable in the diagonal; since
\[
\langle \langle (M, M') \rangle \rangle_t = \langle \langle M \rangle \rangle_t + \langle \langle M' \rangle \rangle_t,
\] we deduce from the Cauchy-Schwartz inequality that $(M, M')$ is in $\mathcal{E}_{\lambda/2}$. By regularizing $f$, we can apply Theorem 2.1.3 and deduce that $(M, M')$ lives in the diagonal. \hfill \square

**Remark.** If $M$ and $M'$ take their values in an open subset $\Delta$ of $V$ such that any two points of $\Delta$ are linked by at least one minimizing $\Delta$-valued geodesic segment, then in previous theorem, the constant numbers $K$ and $R$ refer to $\Delta$ rather than $V$. If moreover this geodesic segment is unique, one can take $R = \infty$.

**Example.** A geodesic ball $B(O, \rho)$ is said to be regular (see [9]) if $B \times B$ does not meet the cut locus and $\rho < \pi/(2\sqrt{K})$. By applying Proposition 2.1.2 to the function $x \mapsto \cos(\sqrt{K}\delta(O, x))$, we deduce from Theorem 2.2.1 that two martingales taking their values in a regular geodesic ball and converging to the same variable coincide: this is the uniqueness result of [9].

We now verify that the part depending on $R$ in the constant $\lambda$ of Theorem 2.2.1 cannot be improved; in particular for manifolds satisfying $K \leq \pi^2/R^2$, this constant appears to be the best one.

**Proposition 2.2.2.** Suppose that $\lambda$ is a constant satisfying $0 < \lambda < \frac{1}{2} \left( \frac{\pi}{\sqrt{2}} \right)^2$. If on our probability space, there exists a one-dimensional $\mathcal{F}_t$ Wiener process, then one can find two different martingales of $\mathcal{E}_\lambda$ converging to the same variable.

**Proof.** From the definition of the injectivity radius, one can deduce that there exist two different geodesic lines $c(t)$ and $\tau(t)$, $0 \leq t \leq 1$, of length less than $\pi/\sqrt{2\lambda}$ and satisfying $c(0) = \tau(0)$, $c(1) = \tau(1)$. Then if $W_t$ is the Wiener process and if
\[
\tau = \inf \{ t \geq 0, \quad |W_t| \geq 1/2 \},
\]
we consider the martingales
\[ M_t = \tau \left( \frac{1}{2} + W_{t\wedge \tau} \right), \quad \overline{M}_t = \tau \left( \frac{1}{2} + W_{t\wedge \tau} \right). \] (2.2.6)

Then these two martingales converge to the same variable and their quadratic variations are at most \( \alpha \pi^2 \tau / (2\lambda) \) for some \( \alpha < 1 \). But from Itô’s formula, the process 
\[ \cos(\sqrt{\alpha}W_t) \exp(\alpha t^2 / 2) \] is a local martingale; when stopped at \( \tau \), it is positive, so it is a supermartingale and in particular
\[ \cos(\sqrt{\alpha}W_t) \exp(\alpha t^2 / 2) \leq 1. \] (2.2.7)

Thus \( M \) and \( \overline{M} \) are in \( E_\lambda \).

Remark. The manifold \( V \) and its universal cover \( \tilde{V} \) have the same sectional curvatures but the injectivity radius may be larger for the cover; if the sectional curvatures are non-positive, the injectivity radius of the cover is infinite, so we can take for \( \lambda \) any positive constant.
Proof. We will denote by $Y_t^\alpha$ the family of processes going from $M_t$ to $M_t$. Consider the cover $\tilde{V}$, the projection $\Phi : \tilde{V} \to V$ and choose a point $X_0$ in $\tilde{V}$ such that $\Phi(X_0) = M_0$; then consider the lifting $X_0^\alpha$ of the process $Y_t^\alpha$ so that $X_0^\alpha = X_0$; then for any $\alpha$, $X_\alpha^\alpha$ is an adapted process and $X_\alpha^\alpha$ does not depend on $\alpha$; moreover $\Phi(X_0^\alpha) = M_t$ and $\Phi(X_1^\alpha) = M_t$ so $X_0^\alpha$ and $X_1^\alpha$ are martingales of $\tilde{V}$ converging to the same variable; since $\Phi$ is locally isometric, they have the same quadratic variation as $M_t$ and $M_t$, so they are in $E_\lambda$. From our assumption, $X_0 = X_1$, so by projection on $V$, $M = M$. \(\square\)

We now prove a result where the homotopy condition is replaced by differentiability. More precisely, we will limit ourselves to the case of a Wiener probability space and will consider martingales which are smooth enough in the sense of the differential calculus on this Wiener space (see §1.3).

**Theorem 2.2.4.** Let $\lambda$ be a non-negative function, let $\tilde{V}$ be the universal cover of $V$ and suppose that the uniqueness holds for $\tilde{V}$-valued martingales of $E_\lambda$. Let $M_t$ and $\overline{M}_t$ be two $V$-valued martingales of $E_\lambda$ converging to the same variable; suppose that $M$ and $\overline{M}$ are in $D^*(V)$. Then $M = \overline{M}$.

The first step in the proof of this theorem is the

**Lemma 2.2.5.** Let $\Psi$ be a real-valued function defined on $C([0, \infty), V)$ which is bounded and uniformly Lipschitz for the sup norm. If $X_t$ is a continuous process in $D^*(V)$, then $\Psi(X)$ is in $D(\mathbb{R})$.

**Proof.** If $x(t)$ is a step function such that $\delta(x(t^-), x(t)) < R$, we can consider the continuous function $y(t)$ obtained by geodesic interpolation and put $\Psi_0(x) = \Psi(y)$; since the sectional curvatures are bounded above by $K$, we can check that $\Psi_0$ is uniformly Lipschitz on

$$\left\{ \sup_t \delta(x(t^-), x(t)) < R \wedge \frac{\pi}{2\sqrt{K}} \right\}.$$ 

Then we can construct a function $\Psi_1$ defined on the set of step functions, which is uniformly Lipschitz and such that $\Psi_1 = \Psi_0$ on

$$\left\{ \sup_t \delta(x(t^-), x(t)) < \frac{R}{2} \wedge \frac{\pi}{4\sqrt{K}} \right\}.$$ 

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Now consider a sequence of step processes $Y^n_t$ converging to $X_t$ and such that (1.3.6) holds; then $\Psi_1(Y^n)$ converges in probability to $\Psi(X)$. Moreover, by embedding $V$ in a Euclidean space, $\Psi_1(Y^n)$ is a Lipschitz function of a finite number of variables of $\mathcal{D}(\mathbb{R})$, so it is in $\mathcal{D}(\mathbb{R})$, and by applying (1.3.3) and (1.3.6), it appears that $D\Psi_1(Y^n)$ is bounded in $L^2$, so from Lemma 1.3.2, $\Psi(X)$ is in $\mathcal{D}(\mathbb{R})$. □

Proof of Theorem 2.2.4. Let $\Phi$ be the projection of $\tilde{V}$ on $V$; note that if $\Phi(x) = \Phi(\pi)$, then either $x = \pi$ or $\delta(x, \pi) \geq 2R$. Consider a point $X_0$ in $\tilde{V}$ such that $\Phi(X_0) = M_0$ and lift the process $\tilde{M}_t$ in order to obtain a $\tilde{V}$-valued martingale $\tilde{X}_t$; similarly let $\tilde{X}_0$ be a point in $\tilde{V}$ such that $\Phi(\tilde{X}_0) = \tilde{M}_0$ and let $\tilde{X}_t$ be the corresponding lifting of $\tilde{M}_t$. We deduce from Lemma 2.2.5 applied to $(\tilde{M}, \tilde{M})$ that for any bounded Lipschitz function $\psi$ on $\tilde{V} \times \tilde{V}$, the variable $\psi(\tilde{X}_\infty, \tilde{X}_\infty)$ is in $\mathcal{D}(\mathbb{R})$; in particular, the variable $\delta(X_\infty, \tilde{X}_\infty) \wedge (2R)$ is in $\mathcal{D}(\mathbb{R})$; but since $\Phi(X_\infty) = \Phi(\tilde{X}_\infty)$, this variable takes its values in $\{0, 2R\}$, so from Lemma 1.3.1 it is deterministic. If it is zero, then $X_\infty = \tilde{X}_\infty$ so from our assumption, $X = \tilde{X}$ and therefore $M = \tilde{M}$. Thus if we assume that $M$ and $\tilde{M}$ are not almost surely equal, then $\delta(X_\infty, \tilde{X}_\infty) \geq 2R$ for any choice of $(X_0, \tilde{X}_0)$; in particular, if we denote by $\mathcal{L}(X_0, \tilde{X}_0; X_\infty, \tilde{X}_\infty)$ the law of $(X_\infty, \tilde{X}_\infty)$ corresponding to an admissible choice of $(X_0, \tilde{X}_0)$ and if we consider the measure

$$m = \sum_{(X_0, \tilde{X}_0)} \mathcal{L}(X_0, \tilde{X}_0; X_\infty, \tilde{X}_\infty)$$

(2.2.8)

then for any measurable subset $A$ of $\tilde{V}$ of diameter less than $2R$, we have $m(A \times A) = 0$. On the other hand since the diameter of $A$ is less than $2R$, we also have

$$m(A \times A) = \text{IP}[M_\infty \in \Phi(A)]$$

(2.2.9)

so it cannot be zero for any $A$ of small diameter. Thus the assumption $M \neq \tilde{M}$ implies a contradiction. □

§2.3 A stability result

We have seen in last subsection that under some conditions, a martingale $M_t$ is determined by its limit $M_\infty$. The aim of this subsection is to estimate the perturbation on $M_t$ induced by a perturbation on $M_\infty$. 19
Theorem 2.3.1. For $x \in V$, let $\kappa(x)$ be the smallest non-negative number dominating the sectional curvatures at $x$. Let $M^\alpha_t$, $\alpha \in \mathbb{R}$ be a measurable family of $V$-valued martingales converging to $M^\infty_t$ such that $\langle \langle M^\alpha \rangle \rangle_\infty$ is bounded by a constant number; put $M_t = M^0_t$.

We suppose that the variable

$$
\sup_{\alpha < \beta} \frac{\delta(M^\alpha_t, M^\beta_t) \wedge 1}{|\beta - \alpha|}
$$

is integrable. Define

$$
Z_t = \limsup_{\alpha, \beta \to 0} \frac{\delta(M^\alpha_t, M^\beta_t)}{|\beta - \alpha|}.
$$

(2.3.1)

Then the process

$$
S_t = Z_t \exp \left( \frac{1}{2} \int_0^t \kappa(M_s) d\langle \langle M \rangle \rangle_s \right)
$$

(2.3.2)

is a uniformly integrable submartingale.

Proof. Let $\psi$ be a nondecreasing function defined on $[0, \infty)$ such that $\psi(z) = z$ if $z$ is small enough, $\psi'(z) = 0$ if $z$ is large enough and $f = \psi \circ \delta$ is smooth. We deduce from the calculation of Lemma 1.1.1 that $f'' + 2\lambda f \geq 0$ for some bounded continuous non-negative function $\lambda$ such that $\lambda(x, x) = \kappa(x)/4$. Thus for any $\alpha < \beta$, the process

$$
S^\alpha_\beta_t = \frac{f(M^\alpha_t, M^\beta_t)}{\beta - \alpha} \exp \left( \int_0^t \lambda(M^\alpha_s, M^\beta_s)(d\langle \langle M^\alpha \rangle \rangle_s + d\langle \langle M^\beta \rangle \rangle_s) \right)
$$

(2.3.3)

is a bounded submartingale. We deduce that the family of variables $\sup_t S^\alpha_\beta_t$ is bounded in probability so in particular, we obtain the uniform convergence in probability of $M^\alpha_t$ to $M_t$ as $\alpha \to 0$. Consider $V$ as a submanifold of a Euclidean space; then the family $M^\alpha_t$ becomes a family of Euclidean semimartingales; if we decompose it, its quadratic variation is uniformly bounded and the variation of its bounded variation part is bounded in probability; thus (apply for instance Corollary 2.2.5 of [13]) the uniform convergence in probability of $M^\alpha_t$ to $M_t$ implies the uniform convergence in probability of $\langle \langle M^\alpha \rangle \rangle_t$ to $\langle \langle M \rangle \rangle_t$. Since $\lambda(M^\alpha_t, M^\beta_t)$ converges to $\kappa(M_t)/4$, we can also get (apply Theorem 2.4.4 of [13]) the convergence in probability of the integral in (2.3.3) to the integral in (2.3.2) as $(\alpha, \beta) \to (0, 0)$. There exists a sequence $(\alpha_k, \beta_k)$ such that the convergence holds almost surely, so

$$
S_t = \limsup_k S^{\alpha_k}_\beta_k.
$$

(2.3.4)

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The processes $S_{t}^{\alpha,\beta}$ are submartingales which are dominated by a uniformly integrable process; we can conclude by means of Fatou’s lemma. □

In this theorem, we have let $\alpha, \beta$ tend to $0$, but of course we can let them tend to another value; thus we obtain the almost sure Lipschitz continuity of $M_{t}^{\alpha}$ with respect to $\alpha$ and estimate $\partial M_{t}^{\alpha}/\partial \alpha$; this process can be viewed as a stochastic Jacobi field along $M_{t}^{\alpha}$.

3. The existence in the differentiable case

In this section we are given a random variable $L$ and we want to construct a $V$-valued martingale converging to this variable. Henceforth we will limit ourselves to the case of a Wiener probability space and in this section, we will consider the case of a differentiable variable $L$ such that $DL$ satisfies some estimate.

§3.1 Statement of the result

The aim of this section is to prove the

\textbf{Theorem 3.1.1.} Suppose that $\Omega$ is the $m$-dimensional Wiener space. Let $\rho_t$ be a deterministic non-negative function satisfying

$$\int_{0}^{\infty} \rho_{t}^{2} dt < 1/K \quad (3.1.1)$$

and let $L$ be a variable of $\mathcal{D}(V)$ satisfying

$$\left( \sum_{j=1}^{m} \mathbb{E} \left[ \left| D_{t}^{j} L \right| \mathcal{F}_{t} \right]^{2} \right)^{1/2} \leq \rho_{t}. \quad (3.1.2)$$

Then there exists a martingale $M_{t}$ converging to $L$ and such that

$$\frac{d}{dt} \langle M \rangle_{t} \leq \frac{\rho_{t}^{2}}{1 - K \int_{t}^{\infty} \rho_{s}^{2} ds}, \quad (3.1.3)$$

Moreover $M$ is in $\mathcal{D}^{*}(V)$.

Remark 1. If the sectional curvatures are non-positive ($K = 0$), condition (3.1.1) simply means that $\rho_{t}$ is square integrable. In the general case, if $\rho_{t}$ is square integrable but
(3.1.1) does not hold, if \( t_0 \) is such that \( \int_{t_0}^{\infty} \rho_t^2 dt = 1/K \), we deduce from Theorem 3.1.1 the existence of a martingale \( M_t, t > t_0 \) converging to \( L \), but do not know whether it can be extended at and before \( t_0 \).

**Remark 2.** We deduce from (3.1.3) that \( \langle \langle M \rangle \rangle_\infty \) is uniformly bounded so \( M_t \) is in the uniqueness class of Theorem 2.2.1.

**Remark 3.** When one is given a random variable \( L \), in order to apply Theorem 3.1.1, one has to find a Wiener process with respect to which \( L \) is measurable; to this end, one has sometimes to enlarge the probability space. The choice of the Wiener process is not unique and different choices may have different behaviours for the application of the theorem.

**Remark 4.** The above results can also be applied when the time interval is the whole real line \( [-\infty, +\infty] \), so that we obtain martingales converging as \( t \to -\infty \) as soon as \( \int_{-\infty}^{+\infty} \rho_t^2 dt < 1/K \). On the other hand, consider the case of the sphere \( S^d \) and let \( M_t \) be the Brownian motion on \( S^d \) stopped at some fixed deterministic time \( T \); then \( M_t \) has no limit as \( t \to -\infty \). Use the canonical isometric embedding of \( S^d \) in \( \mathbb{R}^{d+1} \); then \( M_t \) can be defined on a Wiener space of dimension \( d + 1 \) by the stochastic differential equation

\[
dM_t = (I - M_t M_t^*) dW_t - \frac{d}{2} M_t dt
\]

(3.1.4)

for \( -\infty < t < \infty \). One can check that \( M_T \) is in \( D(\mathbb{R}^{d+1}) \) for \( d \geq 3 \) and that

\[
D_t M_T (D_t M_T)^* = (I - M_T M_T^*) \exp \left( -2 \int_{t}^{T} M_s^* dW_s - d(T - t) \right)
\]

(3.1.5)

for \( t \leq T \) (compare the equations satisfied by both sides as \( T \) varies); by taking the trace and by integrating, we obtain

\[
\sum_j \mathbb{E} \left[ |D_t^j M_T|^2 \right | \mathcal{F}_t] = d \exp \{-(d-2)(T-t)\}.
\]

Thus we can take for \( \rho_t \) the square root of the right-hand side and \( \int \rho_t^2 dt \) is \( d/(d-2) \); it is finite but is more than \( 1/K = 1 \); we also see that the constant \( 1/K \) in Theorem 3.1.1 cannot be improved (at least without taking into account the dimension of the manifold).
The theorem will be proved in several steps. First note that by a deterministic change of time, we can always assume that $L$ is $F_1$ measurable and that $\rho_1$, $0 \leq t \leq 1$ is a constant $\rho < 1/\sqrt{K}$; then we have to find a martingale satisfying
\[
\frac{d}{dt} \langle M \rangle_t \leq \frac{\rho^2}{1 - K \rho^2 (1 - t)}.
\] (3.1.7)
In the proof we will suppose that we are in this simplified framework.

The basic idea is to construct a sequence of step processes which will be proved to converge to our martingale. Let us now define this sequence. Let $\mu_0 < R \wedge \frac{\pi}{\sqrt{K}}$. Then there exists a constant $0 < \mu < \mu_0$ and a nondecreasing smooth function $\psi$ from $I \mathbb{R}^+$ into $I \mathbb{R}^+$ such that
\[
\psi'(z) \leq \frac{\sin(\sqrt{K} z)}{\sqrt{K}}, \quad |\psi''(z)| \leq 1
\] (3.1.8)
for any $z$, $\psi'(z) = 0$ for $z \geq \mu_0$ and
\[
\psi(z) = \frac{1 - \cos(\sqrt{K} z)}{K}
\] (3.1.9)
if $z \leq \mu$, where this expression is by convention $z^2/2$ if $K = 0$. Define also $g = \psi \circ \delta$; we have constructed a smooth function on $V \times V$ which is equivalent to $\delta^2(x)/2$ as $x$ tends to the diagonal. Let $(t^n_i, 1 \leq i \leq n)$ be a sequence of subdivisions of $[0, 1]$ such that the step size $\varepsilon_n = \sup |t^n_{i+1} - t^n_i|$ converges to 0 as $n \to \infty$. Put also $X^n_0 = L$; assuming that $X^n_{i+1}$ is given and $F^n_{t^n_{i+1}}$ measurable, let us construct the variable $X^n_i$. This construction will be based on an approximation of the construction of the Riemannian centre of mass described in [8]. The random function
\[
x \mapsto \mathbb{E}[g(x, X^n_{i+1}) \mid F^n_{t^n_{i}}]
\]
is non-negative, smooth, bounded by $\psi(\infty)$; moreover, if $V$ is not compact, this function converges almost surely to $\psi(\infty)$ as $x \to \infty$; thus it takes its minimal value at at least one point and we can find a $F^n_{t^n_{i}}$ measurable variable $X^n_i$ satisfying
\[
\mathbb{E}[g(X^n_i, X^n_{i+1}) \mid F^n_{t^n_{i}}] = \inf_x \mathbb{E}[g(x, X^n_{i+1}) \mid F^n_{t^n_{i}}].
\] (3.1.10)
Then we can consider the sequence of step processes $M^n_t$ taking the value $X^n_i$ for $t^n_i \leq t < t^n_{i+1}$. The martingale $M_t$ will be the uniform limit in probability of the processes $M^n_t$.
3.2 Preliminary properties

We first give a geometric result about the derivatives of the function $g$.

**Lemma 3.2.1.** Let $x = (x_0, x_1)$ be a point of $V \times V$, $u_0 \in T_{x_0}V$ and $u_1 \in T_{x_1}V$. Then

$$g''_{00}(x)(u_0, u_0) \geq (1 - Kg(x) - Cg^2(x))|u_0|^2$$

for some $C > 0$ and

$$|g''_{01}(x)(u_0, u_1)| \leq |u_0| |u_1|.$$  \(3.2.1\)

**Proof.** We only need to prove the result on \(\{0 < \delta(x) < \mu_0\}\). Let us first prove (3.2.1). Use the decomposition $u_0 = v_0 + w_0$ into parallel and orthogonal vectors; we deduce from Lemma 1.2.1 that

$$g''_{00}(x)(u_0, u_0) \geq (\psi'' \circ \delta)(x)|v_0|^2 + (\psi' \circ \delta)(x)\sqrt{K} \cot(\sqrt{K}\delta(x))|w_0|^2.$$  \(3.2.3\)

In particular this expression is bounded below by some $-C_0|v_0|^2$ so there exists a $C > 0$ such that

$$g''_{00}(x)(u_0, u_0) \geq (1 - Cg^2(x))|u_0|^2$$  \(3.2.4\)

if $\delta(x) \geq \mu$; on the other hand, if $0 < \delta(x) < \mu$, (3.2.3) can be written as

$$g''_{00}(x)(u_0, u_0) \geq (1 - Kg(x))|u_0|^2.$$  \(3.2.5\)

The estimate (3.2.1) is then deduced from (3.2.4) and (3.2.5). Then, in order to prove (3.2.2), we apply

$$g'_0(x)(u_0, u_1) = (\psi'' \circ \delta)(x)\delta'_0(x)(u_0)\delta'_1(x)(u_1) + (\psi' \circ \delta)(x)\delta''_{01}(x)(u_0, u_1)$$  \(3.2.6\)

and we deduce from Lemma 1.2.1 and (3.1.8) that

$$|g'_0(x)(u_0, u_1)| \leq |v_0| |v_1| + |w_0| |w_1|$$

$$\leq |u_0| |u_1|.$$  \(3.2.7\)

$\square$
Lemma 3.2.2. Let $Q$ be a probability measure on $V$; then the function

$$Q \mapsto \inf_x \int g(x,z)Q(dz)$$

is continuous (for the weak convergence on probability measures). When this function is less than some constant positive number, the infimum is obtained at exactly one point which depends continuously on $Q$.

Remark 1. In particular, this result can be applied to the definition (3.1.10) of $X^n_{i+1}$ by taking for $Q$ the conditional law of $X^n_{i+1}$ given $\mathcal{F}^n_{i+1}$.

Remark 2. From the Bienaymé-Chebychev inequality, the second part of the lemma holds in particular when $\int \int g(x,z)Q(dx)Q(dz)$ is less than some constant number.

Proof. Since $g$ is uniformly continuous, the map $Q \mapsto \int g(x,z)Q(dz)$ is continuous uniformly in $x$; thus the map $Q \mapsto \inf_x \int g(x,z)Q(dz)$ is continuous. Then fix $Q$ and note that if $V$ is not compact the function $\gamma : x \mapsto \int g(x,z)Q(dz)$ tends to its supremum as $x \to \infty$ so its infimum is obtained at at least one point. From (3.2.1), $\gamma$ is strictly convex on the set where it is less than some constant number; thus, since $\gamma$ is uniformly Lipschitz, there exists some constant numbers $C_1$ and $C_2$ such that $\gamma$ is strictly convex on the ball $B(x,C_1)$ as soon as $\gamma(x) < C_2$. Moreover, from the Bienaymé-Chebychev inequality, there exists some $C_3 < C_2$ such that $\gamma(x) < C_3$ implies

$$Q[B(x,C_1/2)] > 1/2. \quad (3.2.8)$$

Now if $x_0$ and $x_1$ are two points where the function $\gamma$ gets its minimal value and if this value is less than $C_3$, then from (3.2.8), the two balls $B(x_0,C_1/2)$ and $B(x_1,C_1/2)$ must intersect, so $\delta(x_0,x_1)$ is less than $C_1$; but since $\gamma$ is strictly convex on $B(x_0,C_1)$, we necessarily have $x_0 = x_1$ so the infimum is obtained at exactly one point. The continuous dependence of this point follows easily. \hfill \square

Now, assuming that $X^n_{i+1}$ is in $\mathcal{D}(V)$ and satisfies some estimate, we are going to estimate the distance between $X^n_i$ and $X^n_{i+1}$ so that one can apply Lemma 3.2.2.
Lemma 3.2.3. Suppose that \( X_{i+1}^n \) is in \( D(V) \) and that

\[
\sum_{j=1}^{m} \int_{t_{i}^{n}}^{t_{i+1}^{n}} \mathbb{E} \left[ |D_j^i X_{i+1}^n| \right] dW_j \leq \nu_i^n
\]

(3.2.9)

almost surely for some deterministic number \( \nu_i^n \). There exists a constant number \( C > 0 \) such that

\[
\mathbb{E} \left[ g(X_i^n, X_{i+1}^n) \right] \leq \frac{\nu_i^n}{2} + C(\nu_i^n)^2
\]

(3.2.10)

and

\[
\mathbb{E} \left[ g^2(X_i^n, X_{i+1}^n) \right] \leq C(\nu_i^n)^2.
\]

(3.2.11)

Proof. The basic tool in the proof of this result is the following formula: if \( h(x_0, x_1) \) is a smooth real Lipschitz function on \( V \times V \), if \( \tau \leq t_{i+1}^{n} \) is a stopping time and if \( X \) is a \( \mathcal{F}_\tau \) measurable \( V \)-valued variable, then

\[
h(X, X_{i+1}^n) = \mathbb{E} \left[ h(X, X_{i+1}^n) \right] + \sum_{j=1}^{m} \int_{\tau}^{t_{i+1}^{n}} \mathbb{E} \left[ h'_i(X, X_{i+1}^n)(D_j^i X_{i+1}^n) \right] dW_j.
\]

(3.2.12)

This formula is a consequence of (1.3.4) (first consider the case \( h(x_0, x_1) = h_0(x_0)h_1(x_1) \) and then use classical approximation techniques). In particular, (3.2.12) enables to estimate the conditional variance of \( h(X, X_{i+1}^n) \) given \( \mathcal{F}_\tau \). Since we are on a Wiener space, one can find a time-continuous version for the conditional law of \( X_{i+1}^n \) given \( \mathcal{F}_\tau \); on the other hand we have seen in Lemma 3.2.2 that the map \( Q \mapsto \inf_x \int g(x, z)Q(dz) \) is continuous; from these two remarks, the process \( \inf_x \mathbb{E}[g(x, X_{i+1}^n)|\mathcal{F}_\tau] \) is continuous. Note that it takes the value \( \mathbb{E}[g(X_i^n, X_{i+1}^n)|\mathcal{F}_\tau] \) at \( t = t_i^n \) and the value 0 at time \( t = t_{i+1}^{n} \). Now consider the event

\[
A = \left\{ \mathbb{E} \left[ g(X_i^n, X_{i+1}^n) \right] \geq \psi(\mu)/2 \right\}.
\]

(3.2.13)

\( A \)From the above continuity, on \( A \) there exist a stopping \( t_i^n \leq \tau \leq t_{i+1}^{n} \) and a \( \mathcal{F}_\tau \) measurable variable \( X \) such that

\[
\mathbb{E} \left[ g(X, X_{i+1}^n) \right] = \psi(\mu)/2
\]

(3.2.14)
and \( X \) is a variable which minimizes this expression among \( \mathcal{F}_\tau \) measurable variables. By applying (3.2.12) to the variable \( g(X, X_{n+1}^n) \), we deduce from (3.2.9) that on \( A \),

\[
g(X, X_{n+1}^n) = \psi(\mu)/2 + O(\sqrt{\nu^n}) \tag{3.2.15}
\]

in the spaces \( L^q \) for the probability \( \mathbb{P} \) conditioned on \( \mathcal{F}_{t^n} \). On the other hand apply (3.2.12) to \( g'(0)(X, X_{n+1}^n) \); its conditional mean is zero (since \( X \) is solution of a variational problem), so we deduce that

\[
g'(0)(X, X_{n+1}^n) = O(\sqrt{\nu^n}). \tag{3.2.16}
\]

The sets \( \{g(x) = \psi(\mu)/2\} \) and \( \{g'(0)(x) = 0\} \) are disjoint, so if \( \nu^n \) is less than some constant number, there is a contradiction between (3.2.15) and (3.2.16), so \( \mathbb{P}[A] = 0 \) and therefore

\[
\mathbb{E}[g(X_{n+1}^n, X_{n+1}^n) \mid \mathcal{F}_{t^n}] < \psi(\mu)/2 \tag{3.2.17}
\]

almost surely. By applying (3.2.12) to \( g(X_{n+1}^n, X_{n+1}^n) \) with \( \tau = t^n \), we obtain

\[
\mathbb{P}[\delta(X_{n+1}^n, X_{n+1}^n) \geq \mu \mid \mathcal{F}_{t^n}] = \mathbb{P}[g(X_{n+1}^n, X_{n+1}^n) \geq \psi(\mu) \mid \mathcal{F}_{t^n}]
\]

\[
\leq \mathbb{P}[g(X_{n+1}^n, X_{n+1}^n) - \mathbb{E}[g(X_{n+1}^n, X_{n+1}^n) \mid \mathcal{F}_{t^n}] \geq \psi(\mu)/2 \mid \mathcal{F}_{t^n}]
\]

\[
\leq C(\nu^n)^2. \tag{3.2.18}
\]

Apply also (3.2.12) to \( g'(0)(X_{n+1}^n, X_{n+1}^n) \); from (3.2.2) we get

\[
\mathbb{E}[\left| g'(0)(X_{n+1}^n, X_{n+1}^n) \right|^2 \mid \mathcal{F}_{t^n}] \leq \nu^n \tag{3.2.19}
\]

and

\[
\mathbb{E}[\left| g'(0)(X_{n+1}^n, X_{n+1}^n) \right|^4 \mid \mathcal{F}_{t^n}] \leq C(\nu^n)^2. \tag{3.2.20}
\]

On \( \{\delta(x) < \mu\} \), we have

\[
\left| g'(0)(x) \right|^2 = \left| \psi'(\delta(x)) \right|^2 = \frac{\sin^2(\sqrt{K}\delta(x))}{K} \tag{3.2.21}
\]

and

\[
g(x) = \frac{2}{K} \sin^2\left(\frac{\sqrt{K}\delta(x)}{2}\right) \leq \frac{1}{2} \left| g'(0)(x) \right|^2 + C\left| g'(0)(x) \right|^4. \tag{3.2.22}
\]
Thus
\[
\mathbb{E}[g(X^n_i, X^n_{i+1}) \mid \mathcal{F}^n_i] \leq \frac{1}{2} \mathbb{E}\left[|g'_0(X^n_i, X^n_{i+1})|^2 \mid \mathcal{F}^n_i\right] + C \mathbb{E}\left[|g'_0(X^n_i, X^n_{i+1})|^4 \mid \mathcal{F}^n_i\right]
\]
\[
+ C \mathbb{P}\left[\delta(X^n_i, X^n_{i+1}) \geq \mu \mid \mathcal{F}^n_i\right] \leq \frac{\nu^n_i}{2} + C(\nu^n_i)^2
\]
from (3.2.18), (3.2.19) and (3.2.20). The estimate (3.2.11) is proved similarly by noticing that \( g(x) \leq C|g'_0(x)|^2 \) on \( \{\delta(x) < \mu\} \).

§3.3 A stochastic implicit function theorem

From Lemmas 3.2.2 and 3.2.3, we know that if \( X^n_{i+1} \) is in \( \mathcal{D}(V) \) and \( DX^n_{i+1} \) is not too large, then \( X^n_i \) is uniquely defined by (3.1.10) and the distance between \( X^n_i \) and \( X^n_{i+1} \) is estimated by (3.2.10) and (3.2.11); moreover
\[
\mathbb{E}[g'_0(X^n_i, X^n_{i+1}) \mid \mathcal{F}^n_i] = 0
\]
and \( X^n_i \) depends continuously on the conditional law of \( X^n_{i+1} \) given \( \mathcal{F}^n_i \). We still have to prove that \( X^n_i \) is in \( \mathcal{D}(V) \) and to estimate \( DX^n_i \); to this end, we need a stochastic implicit function theorem.

**Lemma 3.3.1.** Suppose that \( X^n_{i+1} \) is in \( \mathcal{D}(V) \). If \( \mathbb{E}[g(X^n_i, X^n_{i+1}) \mid \mathcal{F}^n_i] \) is uniformly less than some constant number, then \( X^n_i \) is in \( \mathcal{D}(V) \).

**Proof.** Let \( k \) be an integer; from Lemma 3.2.2, we know that there exists a constant \( C_0 > 0 \) (not depending on \( k \)) such that on the subset
\[
A_{k}^{C_0} = \left\{(y_1, \ldots, y_k); \frac{1}{k^2} \sum_{j \neq j} g(y_j, y_l) < C_0\right\}
\]
of \( V^k \), the relation
\[
\sum_{j} g(\beta_0(y_1, \ldots, y_k), y_j) = \inf_{x \in V} \sum_{j} g(x, y_j)
\]
defines a unique point \( \beta_0(y_1, \ldots, y_k) \). Moreover from the classical implicit function theorem, and using the estimates of Lemma 3.2.1, this function is smooth and locally
\[
\delta(\beta_0(y), \beta_0(y)) \leq C \sum_{j=1}^{k} \delta(y_j, y_j)
\]
where \( C \) is a constant number which does not depend on \( k \). Now let \( \phi \) be a smooth real function defined on \([0, \infty)\) such that \( \phi(z) = 1 \) if \( z \leq C_0/2 \), \( \phi(z) = 0 \) if \( z \geq C_0 \); consider \( V \) as a submanifold of some Euclidean space \( \mathbb{R}^r \) and define on \( V^k \) the \( \mathbb{R}^r \)-valued function

\[
\beta(y) = \beta_0(y) \phi \left( \frac{1}{k^2} \sum_{j,l} g(y_j, y_l) \right) + \frac{1}{k} \sum_j y_j \left( 1 - \phi \left( \frac{1}{k^2} \sum_{j,l} g(y_j, y_l) \right) \right). \tag{3.3.5}
\]

Then

\[
|\beta(y) - \beta(y)| \leq \frac{C}{k} \sum_{j=1}^k \delta(y_j, y_j) \tag{3.3.6}
\]

for some \( C \) not depending on \( k \); moreover on \( A_k^{C_0/2} \), \( \beta \) is \( V \)-valued and satisfies (3.3.3). Now enlarge the probability space into \( \Omega \times \Omega^\mathbb{N} \) and let \((W_t, W_1^t, \ldots)\) be the canonical process, the components of which are independent standard Wiener processes. If \((\omega, \omega^1, \ldots)\) is a generic element of this space, denote

\[
\omega_j(t) = \omega(t \wedge t_n^i) + \omega_j(t \vee t_n^i) - \omega_j(t_n^i) \tag{3.3.7}
\]

and

\[
Y^k = \beta(X_{i+1}^n(\omega^1), \ldots, X_{i+1}^n(\omega^k)). \tag{3.3.8}
\]

For almost each fixed \((\omega^1, \ldots)\), the variables \( X_{i+1}^n(\omega^j) \) are in \( D(V) \) so, since \( \beta \) satisfies (3.3.6), the variables \( Y^k \) are in \( D(\mathbb{R}^r) \) and

\[
|D_t^j Y^k| \leq \frac{C}{k} \sum_{l=1}^k |D_t^j X_{i+1}^n(\omega^l)| \tag{3.3.9}
\]

for \( t < t_n^i \), so that

\[
\mathbb{E} \left[ \int_0^{t_n^i} |D_t^j Y^k|^2 \, dt \mid W^1, \ldots \right] \leq \frac{C}{k} \sum_{l=1}^k \mathbb{E} \left[ \int_0^{t_n^i} |D_t^j X_{i+1}^n(\omega^l)|^2 \, dt \mid W^1, \ldots \right]. \tag{3.3.10}
\]

Moreover from the law of large numbers and the continuity property of Lemma 3.2.2, \( Y^k \) converges almost surely on \( \Omega \times \Omega^\mathbb{N} \) to \( X_1^n \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^k \mathbb{E} \left[ \int_0^{t_n^i} |D_t^j X_{i+1}^n(\omega^l)|^2 \, dt \mid W^1, \ldots \right] = \mathbb{E} \left[ \int_0^{t_n^i} |D_t^j X_{i+1}^n|^2 \, dt \mid \mathcal{F}_{t_n^i} \right] \tag{3.3.11}
\]

almost surely; so let us fix \((\omega^1, \ldots)\) at a point at which these two convergences hold almost surely on \( \Omega \); from (3.3.10) and (3.3.11), \( DY^k \) is bounded in \( L^2 \) so we deduce from Lemma 1.3.2 that \( X_1^n \) is in \( D(V) \). □
Lemma 3.3.2. Under the assumptions of Lemma 3.3.1,
\[ |D^j_t X^n_i| \leq \left( 1 + K \mathbb{E}[g(X^n_i, X^n_{i+1}) \mid \mathcal{F}_t^n] \right) + C \mathbb{E}[g^2(X^n_i, X^n_{i+1}) \mid \mathcal{F}_t^n] \mathbb{E}[|D^j_t X^n_{i+1}| \mid \mathcal{F}_t^n] \] (3.3.12)
for \( t \leq t^n_i \).

Proof. When we consider \( V \) as a submanifold of \( \mathbb{R}^r \), each tangent space \( T_x V \) becomes canonically isomorphic to a vectorial subspace of \( \mathbb{R}^r \); in particular the function \( h = g'_0 \) can be considered as \( \mathbb{R}^r \)-valued and (3.3.1) can be viewed as an equality in \( \mathbb{R}^r \); since \( X^n_i \) and \( X^n_{i+1} \) are in \( D(V) \) we can differentiate this relation (we apply the properties of the Wiener derivative applied to a conditional expectation) and obtain
\[ \mathbb{E}\left[h'_0(X^n_i, X^n_{i+1})(D^j_t X^n_i) + h'_1(X^n_i, X^n_{i+1})(D^j_t X^n_{i+1}) \mid \mathcal{F}_t^n\right] = 0 \] (3.3.13)
for \( t \leq t^n_i \); this relation is again in \( \mathbb{R}^r \). Now take the scalar product of (3.3.13) with \( D^j_t X^n_i \) which is \( \mathcal{F}_t^n \) measurable; by replacing \( h \) by \( g'_0 \), we get
\[ \mathbb{E}\left[g''_{00}(X^n_i, X^n_{i+1}) (D^j_t X^n_i, D^j_t X^n_i) \mid \mathcal{F}_t^n\right] \]
\[ = -\mathbb{E}\left[g''_{01}(X^n_i, X^n_{i+1}) (D^j_t X^n_i, D^j_t X^n_{i+1}) \mid \mathcal{F}_t^n\right] \] (3.3.14)
\( \square \)

§3.4 End of the proof

We will first prove the weak existence of the martingale; more precisely, let \( C(V) \) be the space of \( V \)-valued continuous functions defined on \([0, 1]\), and consider the product space \( \Omega \times C(V) \) and its canonical process \((W_t, M_t)\); variables defined on \( \Omega \) such as \( L \) can also be considered as variables defined on \( \Omega \times C(V) \).

Lemma 3.4.1. On \( \Omega \times C(V) \) there exists a probability \( \mathbb{P}_0 \) such that if \( \mathcal{G}_t \) is the completed right-continuous filtration generated by \((W_t, M_t)\), then \( W_t \) is a \( \mathcal{G}_t \) Wiener process and \( M_t \) is a \( \mathcal{G}_t \) martingale satisfying \( M_1 = L \) and (3.1.7).

Proof. Suppose that \( X_{i+1}^n \) is in \( D(V) \) and
\[ \left( \sum_{j=1}^{m} \mathbb{E}[|D_t X_{i+1}^n| \mid \mathcal{F}_t]^2 \right)^{1/2} \leq \rho_{i+1}^n \] (4.1.1)
(we have assumed that this condition is satisfied for \(i + 1 = n\) with \(\rho^n_{i+1} = \rho\)). Then (3.2.9) is satisfied with
\[
\nu^n_i = (\rho^n_{i+1})^2 (t^n_{i+1} - t^n_i).
\] (3.4.2)

In particular, from Lemma 3.2.3,
\[
\mathbb{E}[g(X^n_i, X^n_{i+1}) \mid F_{t^n_i}] \leq \frac{(\rho^n_{i+1})^2}{2} (t^n_{i+1} - t^n_i) + C (\rho^n_{i+1})^4 (t^n_{i+1} - t^n_i)^2.
\] (3.4.3)

For any constant number \(C_0\), if \(\rho^n_{i+1} \leq C_0\) and if \(n\) is large enough, then the expression of (3.4.3) is small enough and we can apply the previous lemmas; from Lemmas 3.3.1 and 3.3.2, \(X^n_i\) is in \(D(V)\) and satisfies an estimate similar to (3.4.1) with
\[
\rho^n_i = \rho^n_{i+1} \left[1 + \frac{K}{2} (\rho^n_{i+1})^2 (t^n_{i+1} - t^n_i) + C (\rho^n_{i+1})^4 (t^n_{i+1} - t^n_i)^2 \right].
\] (3.4.4)

By comparing with the ordinary backward equation
\[
\dot{\phi}_s = -\frac{K}{2} \phi^3_s, \quad s \leq 1 \quad \phi_1 = \rho,
\] (3.4.5)

which explodes at time \(1 - 1/(K\rho^2)\) if \(K > 0\), it appears that \(\rho^n_i\) remains asymptotically bounded if the assumption \(\rho < 1/\sqrt{K}\) holds. More precisely, we have
\[
\rho^n_i \leq \frac{\rho}{\sqrt{1 - K\rho^2(1 - t^n_i)}} + o(1)
\] (3.4.6)

where the \(o(1)\) term tends to 0 as \(n \to \infty\) uniformly in \(i\). In particular,
\[
\mathbb{E}[g(X^n_i, X^n_{i+1}) \mid F_{t^n_i}] \leq -\frac{\rho^2 (t^n_{i+1} - t^n_i)}{2(1 - K(1 - t^n_i)\rho^2)} + o(t^n_{i+1} - t^n_i)
\] (3.4.7)

and
\[
\mathbb{E}[g^2(X^n_i, X^n_{i+1}) \mid F_{t^n_i}] \leq C (t^n_{i+1} - t^n_i)^2.
\] (3.4.8)

Now let \(h\) be a bounded smooth function with bounded derivatives defined on \(V\); by applying the Taylor formula to \(h\) at \(X^n_i\) and by using the inverse exponential function at \(X^n_i\) which is defined on a neighbourhood of \(X^n_i\), we obtain
\[
h(X^n_{i+1}) = h(X^n_i) + h'(X^n_i)(\exp^{-1}_{X^n_i} X^n_{i+1})
\]
\[
+ \frac{1}{2} h''(X^n_i)(\exp^{-1}_{X^n_i} X^n_{i+1}, \exp^{-1}_{X^n_i} X^n_{i+1}) + O(|\exp^{-1}_{X^n_i} X^n_{i+1}|^3).
\] (3.4.9)

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Since  
\[ \exp^{-1} X^n_{i+1} = g_0(X^n_i, X^n_{i+1}) + O(g^{3/2}(X^n_i, X^n_{i+1})), \]  
(3.4.10)  
by taking the conditional expectation in (3.4.9), by using (3.3.1) and (3.4.8), we deduce  
\[ \mathbb{E}[h(X^n_{i+1}) \mid \mathcal{F}_{t^n_i}] = h(X^n_i) + \frac{1}{2} \mathbb{E}\left[ h''(X^n_i)(g_0(X^n_i, X^n_{i+1}), g_0(X^n_i, X^n_{i+1})) \mid \mathcal{F}_{t^n_i} \right] 
+ O\left((t^n_{i+1} - t^n_i)^{3/2}\right). \]  
(3.4.11)  

If \( \lambda_h(x) \) is the smallest eigenvalue of \( h''(x) \) and if \( \lambda_h^- = (\lambda_h) \vee 0 \), we deduce  
\[ \mathbb{E}[h(X^n_{i+1}) \mid \mathcal{F}_{t^n_i}] \geq h(X^n_i) + \frac{1}{2} \lambda_h(X^n_i) \mathbb{E}\left[ |g_0'(X^n_i, X^n_{i+1})|^2 \mid \mathcal{F}_{t^n_i} \right] - C(t^n_{i+1} - t^n_i)^{3/2} \]  
\[ \geq h(X^n_i) - C\lambda_h^-(X^n_i)(t^n_{i+1} - t^n_i) - C(t^n_{i+1} - t^n_i)^{3/2}. \]  
(3.4.12)  

Now consider the step process \( M^n_t \) defined in §3.1, which is \( X^n_t \) on the time interval \([t^n_i, t^n_{i+1})\); we want to prove that its law is \( C \)-tight. We deduce from (3.4.12) that  
\[ \mathbb{E}[h(M^n_t) \mid \mathcal{F}_s] \geq h(M^n_s) - C \int_s^t \lambda_h^-(M^n_u)du - C\sqrt{\varepsilon_n}(t - s) \]  
(3.4.13)  
for \( s \leq t \) in the subdivision \( (t^n_i) \) and where we recall that \( \varepsilon_n \) is the step size of the subdivision. On the other hand if in Lemma 3.2.3 we replace \( t^n_i \) by another time \( s \), we see that there exist \( \mathcal{F}_s \) measurable variables \( X^n_{s,t} \) such that  
\[ \mathbb{E}[g^2(X^n_{s,t}, M^n_t) \mid \mathcal{F}_s] \leq C(t - s)^2, \quad \mathbb{E}[g(X^n_{s,t}, M^n_t) \mid \mathcal{F}_s] \leq C(t - s). \]  
(3.4.14)  

By applying (3.4.13) to the function \( x_1 \mapsto g(x_0, x_1) \), we obtain  
\[ \mathbb{E}[g(X^n_{s,t}, M^n_t) - g(X^n_{s,t}, M^n_s) \mid \mathcal{F}_s] \geq -C(t - s) \]  
(3.4.15)  
so by substracting these two estimates,  
\[ g(X^n_{s,t}, M^n_s) \leq C(t - s). \]  
(3.4.16)  

Since \( \delta \wedge 1 \leq C\sqrt{g} \), we deduce from (3.4.14) and (3.4.16) estimates on \( \delta(X^n_{s,t}, M^n_s) \) and \( \delta(X^n_{s,t}, M^n_t) \); from the triangle inequality, we get for \( s \) and \( t \) in the subdivision  
\[ \mathbb{E}[\delta^4(M^n_s, M^n_t) \wedge 1 \mid \mathcal{F}_s] \leq C(t - s)^2. \]  
(3.4.17)
Moreover the law of \( M_t^n = L \) is fixed and therefore tight, so the \( C \)-tightness of the process \( M^n \) is proved. This implies that the law of \((W, M^n)\) is also \( C \)-tight, so let us consider the limit \( \mathbb{P}_0 \) of some subsequence; for any real bounded measurable function \( \phi \) defined on \( \Omega \times C(V) \) which is continuous with respect to the second variable, the mean of \( \phi(W, M^n) \) converges (for the subsequence) to the \( \mathbb{P}_0 \)-expectation of \( \phi(W, M) \) (apply [7]). In particular, since \( M_t^n = L \), we also have \( M_1 = L \); it is also easy to check that \( W_t \) is a \( \mathcal{G}_t \) Wiener process; on the other hand, by multiplying both sides of (3.4.13) by any continuous function of \((W_u, M_u; u \leq s)\), by taking the expectation and by taking the limit as \( n \to \infty \), we prove that

\[
h(M_t) - C \int_0^t \lambda^-_n(M_s) \, ds
\]

is a \( \mathcal{G}_t \) submartingale. Thus, from the Darling definition [2] of martingales, the process \( M_t \) is a \( \mathcal{G}_t \) martingale. Since \( g \sim \delta^2/2 \) in the neighbourhood of the diagonal, the quadratic variation of \( M_t^n \) is estimated from (3.4.7) and by taking the limit, we obtain (3.1.7). \( \square \)

We finish the proof of Theorem 3.1.1 in two steps; we first prove that the process \( M_t \) obtained in Lemma 3.4.1 is adapted to the filtration \( \mathcal{F}_t \) of \( W_t \) so that it can be realized on \( \Omega \); then we check that \( M_t \) is in \( \mathcal{D}^\ast(V) \).

**Proof of the strong existence.** This will follow from the uniqueness theorem of §1; the Watanabe-Yamada method for proving strong existence for stochastic differential equations (see Chapter 8 of [15]) can also be applied in our context. On the space \( \Omega \times C(V) \times C(V) \) with canonical process \((W, M_1, M_2)\), consider the probability \( \mathbb{P}_1 \) defined by

\[
\mathbb{E}_{\mathbb{P}_1} \left[ f(W)\phi_1(M^1)\phi_2(M^2) \right] = \mathbb{E}_{\mathbb{P}_1} \left[ f(W)\mathbb{E}_{\mathbb{P}_0} [\phi_1(M) \mid W] \mathbb{E}_{\mathbb{P}_0} [\phi_2(M) \mid W] \right] \quad (3.4.18)
\]

for bounded measurable functionals \( f, \phi_1, \phi_2 \). Let \( \mathcal{G}_t^1 \) be the filtration generated by the process \((W, M^1, M^2)\); one verifies that under \( \mathbb{P}_1 \), \((W, M^1)\) and \((W, M^2)\) have law \( \mathbb{P}_0 \), that \( W_t \) is a \( \mathcal{G}_t^1 \) Wiener process, that \( M^1 \) and \( M^2 \) are conditionally independent given \( W \), so that \( M_t^1 \) and \( M_t^2 \) are \( \mathcal{G}_t^1 \) martingales with value \( L \) at final time; since they satisfy (3.1.7), we deduce from Theorem 2.2.1 that \( M^1 = M^2 \). Thus conditionally on \( W = \omega \), \( M^1 \) and \( M^2 \) are independent and almost surely equal, so are equal to some \( M(\omega) \). Now \( M \) is measurable with respect to the \( \sigma \)-field of \( W \) and \( M_t \) is independent from future increments

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of $W$, so $M_t$ is $\mathcal{F}_t$ adapted; thus it is a $\mathcal{F}_t$ martingale defined on $\Omega$ and converging to $L$.

Proof of the regularity. By using elementary results concerning weak and stable convergences (see [7]), we can deduce from the previous proof that $M^n$ converges uniformly in probability to $M_t$. Thus in order to prove that $M_t$ is in $D^*(V)$ it is sufficient to prove (1.3.6) for $Y^n = M^n$, that is

$$\limsup_{n \to \infty} \sum_j \mathbb{E} \int_0^1 \sup_i |D^j_t X^n_i|^2 dt < \infty.$$  \hfill (3.4.19)

But we deduce from Lemma 3.3.2 that there exists a constant $C > 0$ such that for $t$ fixed, the process $\exp(Ct^n)|D^j_t X^n_i|$ is a discrete-time submartingale; from Doob’s inequality, the left-hand side of (3.4.19) is estimated by $4e^{2C} \|L\|^2_D$. \hfill \Box

4. The existence on small convex domains

We now prove that if $\Delta$ is an open subset of $V$ satisfying some assumption, then one can construct $\Delta$-valued martingales in the non differentiable case; this can in particular be applied to manifolds with non-positive curvature. As in previous section, we suppose that the probability space is a Wiener space. If $V$ is embedded in a Euclidean space $\mathbb{R}^r$, the spaces $IL^q(\Delta)$ are defined to be the subsets of the spaces $IL^q(\mathbb{R}^r)$ consisting of $\Delta$-valued variables, with the induced topology (this does not depend on the embedding).

Theorem 4.1. Suppose that $\Omega$ is a Wiener probability space. For $x \in V$, let $\kappa(x)$ be the smallest non-negative number dominating the sectional curvatures at $x$. Let $1 < p \leq \infty$ and $1 \leq q < \infty$ be such that $1/p + 1/q = 1$. Let $\Delta$ be an open connected and simply connected subset of $V$ and suppose that there exists a smooth positive function $f$ such that $0 < c_0 < f < c_1$ on $\Delta$, $0 < f < c_0$ on $V \setminus \Delta$ and $f'' + pf \leq 0$ on $\Delta$ (the case $p = \infty$ is possible only when $\kappa \equiv 0$ on $\Delta$). Then there exists a unique continuous function $L \mapsto (M_t)$ defined on $IL^q(\Delta)$, with values in the space of $\Delta$-valued continuous processes endowed with the uniform convergence in probability, and such that

(a) for any $L$, $M_t$ is a martingale converging to $L$;

(b) if $L$ is in $S(\Delta)$, then $\|M\|_\infty$ is bounded.
Example. The regular geodesic balls of [9] satisfy the assumptions of the theorem for some \( p \): if \( O \) is the centre, use the function \( x \mapsto \cos(\sqrt{K_p} \delta(O, x)) \). Thus the existence theorem of [9] can be deduced from Theorem 4.1.

Remark. We have stated the uniqueness of a continuous function \( L \mapsto (M_t) \) satisfying (a) and (b), but we do not know whether \( M_t \) is the unique martingale converging to \( L \).

Proof. First note that one can apply Theorem 2.1.3 to \( \Delta^\beta = \{ f > \beta \} \) for any \( \beta > c_0 \), so that any martingale with bounded quadratic variation converging to a variable in \( \Delta^\beta \) lives in \( \Delta^\beta \); moreover from Proposition 2.1.2, any \( \Delta \)-valued martingale is in \( E_{p_{\kappa/2}} \); more precisely, the variables

\[
\mathbb{E} \left[ \exp \frac{P}{2} \int_t^\infty \kappa(M_s) d\langle M \rangle_s \bigg| \mathcal{F}_t \right]
\]

are uniformly bounded by some constant number. Note also that two points of \( \Delta \) are linked by at least one \( \Delta \)-valued geodesic (one can prove that if \( x_0 \) is a point of \( \Delta \), the set of points which are linked to \( x_0 \) by a \( \Delta \)-valued geodesic segment is both open and closed in \( \Delta \); to this end we use the fact that if the end points of a \( \Sigma \)-valued geodesic are in \( \Delta \), then the geodesic is \( \Delta \)-valued); one can also prove from comparison theorems that there do not exist conjugate points on a \( \Delta \)-valued geodesic; since \( \Delta \) is simply connected, one can deduce from the homotopy lemma (see Lemma 2.6.4 of [10]) that two points of \( \Delta \) are linked by exactly one geodesic; moreover this geodesic depends smoothly on its end points; in particular, \( \Delta \) is diffeomorphic to a star-shaped open subset of \( \mathbb{R}^d \) (use the exponential function at some point), so \( S(\Delta) \) is dense in \( L^q(\Delta) \). After these geometrical preliminaries, let us prove the theorem. First note that the uniqueness is a consequence of Theorem 2.2.1 and of the density of \( S(\Delta) \) in \( L^q(\Delta) \). For the existence, let \( L \) be a variable in \( L^q(\Delta) \); one can find a sequence \( L^n \) of functionals of \( S(\Delta) \) such that \( \delta(L^n, L) \) converges to 0 in \( L^q \) and \( f(L^n) > \beta^n \) for some \( \beta^n > c_0 \); for each \( n \) there exists a \( \Delta \)-valued martingale \( M^n_t \), \( t \geq t_n \), with bounded quadratic variation and converging to \( L^n \); the process \( M^n_t \) is in \( D^*(V) \) and by applying Theorem 2.3.1 to perturbations on the Wiener process, we can check that

\[
S_{n,t} = |D_s M^n_t| \exp \frac{1}{2} \int_s^t \kappa(M^n_u) d\langle M^n \rangle_u,
\]

\( t \geq s \), is a submartingale; moreover for any fixed \( n \), the derivatives \( D_s M^n_{\infty} \) are bounded.
and the variables
\[
\mathbb{E} \left[ \exp \frac{1}{2} \int_t^\infty \kappa(M^n_u) d\langle M^n \rangle_u \middle| \mathcal{F}_t \right]
\]
are bounded so \( D_s M^n_t \) is bounded. Thus \( M^n_t \) can be defined on the whole time interval \((t_n = 0);\) if indeed \( t_n > 0, \) since \( D_s M^n_t \) is bounded, we deduce that \( M^n_t \) can always be extended to an interval \([t_n - C, \infty].\) For each \( m \) and \( n, \) let \( L^{m,n,\alpha}, 0 \leq \alpha \leq 1 \) be the geodesic interpolation between \( L^m \) and \( L^n; \) then \( \alpha \mapsto L^{m,n,\alpha} \) is continuous into \( S(\Delta), \) we can construct the martingales \( M^{m,n,\alpha}_t \) converging to \( L^{m,n,\alpha}, \) apply Theorem 2.3.1 to them and estimate the Jacobi field \( \partial M^{m,n,\alpha}_t / \partial \alpha \) for any \( 0 \leq \alpha \leq 1. \) Since these martingales are bounded in \( \mathcal{E}_{pe/2}, \) we deduce that
\[
\mathbb{P} \left[ \sup_t \delta(M^{m}_t, M^{n}_t) > u \right] \leq \frac{C}{u} \mathbb{E} \left[ \delta(L^m, L^n)^q \right]^{1/q}.
\] (4.2)
Thus \( M^n_t \) has in \( \overline{\Delta} \) a uniform limit in probability and this limit is a martingale converging to \( L; \) from Theorem 2.1.3, \( M_t \) is \( \Delta \)-valued. The continuity of the map \( L \mapsto (M_t) \) also follows from (4.2) which can be extended to any sequence of \( \Delta \)-valued variables \( L^n. \)

The continuity property of Theorem 4.1 can be improved by the

**Proposition 4.2.** Under the assumptions of Theorem 4.1, let \((A, d)\) be a metric space and let \( L^n, \alpha \in A, \) be a family of \( \Delta \)-valued variables such that almost surely, \( \alpha \mapsto L^\alpha \) is uniformly continuous; suppose also that \( \sup_{O} \delta(O, L^\alpha) \) is in \( L^g \) for some (or any) \( O. \) For any \( \alpha, \) let \( M^\alpha_t \) be the martingale converging to \( L^\alpha \) constructed in Theorem 4.1; then there is a version of \((M^\alpha_t)\) which is almost surely continuous on \( A \times [0, \infty]. \)

**Proof.** We deduce from (4.2) that
\[
\delta(M^\alpha_0, M^\beta_0) \leq C \mathbb{E} \left[ \delta(L^\alpha, L^\beta)^q \right]^{1/q}.
\] (4.3)
Similarly, by working conditionally on \( \mathcal{F}_t, \) we can check that
\[
\delta(M^\alpha_t, M^\beta_t) \leq C \mathbb{E} \left[ \delta(L^\alpha, L^\beta)^q \middle| \mathcal{F}_t \right]^{1/q}.
\] (4.4)
Thus
\[
\text{ess sup}_{d(\alpha, \beta) \leq \varepsilon} \sup_t \delta(M^\alpha_t, M^\beta_t) \leq C \sup_t \mathbb{E} \left[ \sup_{d(\alpha, \beta) \leq \varepsilon} \delta(L^\alpha, L^\beta)^q \middle| \mathcal{F}_t \right]^{1/q}
\] (4.5)

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and therefore, by using an inequality for submartingales, for \( \eta > 0 \),

\[
\mathbb{P} \left[ \text{ess sup} \sup_{t \in [0, T]} \delta(M_t^\alpha, M_t^\beta) > \eta \right] \leq \mathbb{P} \left[ \sup_t \mathbb{E} \left[ \sup_{d(\alpha, \beta) \leq \varepsilon} \delta(L_t^\alpha, L_t^\beta)^q \mid \mathcal{F}_t \right] > \frac{\eta^q}{C^q} \right] \\
\leq \frac{C^q}{\eta^q} \mathbb{E} \left[ \sup_{d(\alpha, \beta) \leq \varepsilon} \delta(L_t^\alpha, L_t^\beta)^q \right] \tag{4.6}
\]

Take the limit as \( \varepsilon \downarrow 0 \) on both sides of this inequality; the right-hand side converges to 0 for any \( \eta > 0 \), so we deduce

\[
\lim_{\varepsilon \to 0} \text{ess sup} \sup_{t \in [0, T]} \delta(M_t^\alpha, M_t^\beta) = 0 \tag{4.7}
\]

almost surely. If we restrict ourselves to rational values of \( \alpha \), \( M_t^\alpha \) is almost surely continuous with respect to \( \alpha \) uniformly in \((t, \alpha)\), so it is continuous with respect to \((t, \alpha)\); for real values of \( \alpha \), if we choose for \((M_t^\alpha, 0 \leq t \leq \infty)\) a measurable accumulation point of \((M_t^\beta)\), \( \beta \) rational tending to \( \alpha \), we obtain a version satisfying the condition of the proposition. \( \square \)

If \( \Delta \) is not simply connected, we can consider its universal cover; for instance, if \( V \) is connected and its sectional curvatures are non-positive, then its universal cover \( \tilde{V} \) is a Cartan-Hadamard manifold, so we can apply Theorem 4.1 to \( \Delta = \tilde{V} \) with \( p = \infty \); if \( L \) is a variable of \( \mathcal{L}^1(V) \), we can lift it into a variable of \( \mathcal{L}^1(\tilde{V}) \), construct a \( \tilde{V} \)-valued martingale and by projection, prove the existence of a \( V \)-valued martingale converging to \( L \). If \( V \) is not simply connected, the lifting is not unique and consequently, the martingale is not unique. Actually, we have a more precise existence theorem; by using the notion of homotopy defined in §2.2, we can prove the

Proposition 4.3. Let \( \Delta \) be a connected subset of \( V \) and suppose that its universal cover satisfies the conditions of Theorem 4.1. Let \( X_t \) be a \( \Delta \)-valued \( \mathcal{F}_t \) adapted continuous process converging almost surely to \( L \). Suppose that \( \sup_t \delta(O, X_t) \) is in \( \mathcal{L}^3 \). Then there exists a \( \Delta \)-valued martingale converging to \( L \) and homotopic to \( X \).

Proof. Consider a continuous lifting \( \tilde{X}_t \) of \( X_t \) in \( \tilde{\Delta} \); we can construct a family of \( \tilde{\Delta} \)-valued martingales \( \tilde{Y}_t^\alpha \), \( 0 \leq \alpha \leq \infty \), converging to \( \tilde{X}_\alpha \), and from Proposition 4.2, we can choose a version which is almost surely continuous with respect to \((t, \alpha)\); note that \( \tilde{Y}_t^\alpha = \tilde{X}_\alpha \) for
\[ t \geq \alpha. \] Let \( Y_\alpha^t \) be the projection of \( \tilde{Y}_\alpha^t \) on \( \Delta \) if \( t \leq \alpha \), and put \( Y_\alpha^t = X_t \) for \( t \geq \alpha \); put also \( M_t = Y_\alpha^\infty \); then \( M_t \) is martingale converging to \( L \) and \( Y_\alpha^t \) is a homotopy path from \( X_t \) to \( M_t \).

5. Application to partial differential equations

We want to apply previous results to a smooth Markovian case. If one is given a Markov transition kernel, one has to realize it on a Wiener space; when this is possible, such a realization is not unique. In this section, we suppose that we have constructed a realization satisfying some assumptions and we deduce the existence of a probabilistic solution to the heat equation. Then we will consider the Dirichlet problem.

**Theorem 5.1.** On our Wiener space, consider the stochastic differential equation

\[ dX_t = \beta(t, X_t)dt + \sigma(t, X_t) \circ dW_t \]

on a smooth submanifold \( U \) of some Euclidean space, for smooth coefficients \( \beta \) and \( \sigma \); we suppose that when (5.1) is written in Itô’s form in the Euclidean space, the coefficients are uniformly Lipschitz and have at most linear growth. We denote by \( X_{t,x}^s \), \( s \leq t \leq 0 \) the stochastic flow associated to (5.1). Let \( \phi \) be a \( C^1 \) Lipschitz function from \( U \) into \( V \). We suppose that

\[ \sup_x \mathbb{E}\left| \phi'(X_{0,x}^t) \frac{\partial X_{0,x}^t}{\partial x} \sigma(t, x) \right| \leq \rho_t \]

for some function \( \rho_t \) such that \( \int_{-\infty}^0 \rho_t^2 dt \) is finite and less than \( 1/K \). Then there exists a unique measurable \( V \)-valued function \( f \) defined on \( (-\infty, 0] \times U \) such that \( f(0, .) = \phi \) and for any \((s, x), f(t, X_{t,x}^s), s \leq t \leq 0 \) is a \( V \)-valued martingale with bounded quadratic variation.

**Proof.** It follows from the study of (5.1) (see [12]) that \( \phi(X_{0,x}^s) \) is in \( D(V) \) and that

\[ D_t \phi(X_{0,x}^s) = \phi'(X_{0,x}^t) \frac{\partial X_{0,x}^t}{\partial y} \sigma(t, y) \bigg|_{y=X_{t,x}^s} \]

for \( s \leq t \leq 0 \). Thus from (5.2), we can apply Theorem 3.1.1 and \( f(s, x) \) is necessarily the value at time \( s \) of the martingale constructed in this theorem. More precisely, if \( S = (t_i) \)
is a subdivision of \((-\infty, 0]\), for \(t \geq s\) and \(t_i \leq t < t_{i+1}\), we denote by \(M_t^{S,s,x}\) the point at which the function \(\mathbb{E}[g(x, M_{t_i+1}^{S,s,x}) \mid \mathcal{F}_t]\) takes its minimum: from the study of §3, this point is unique when
\[
\varepsilon(S) = \sup_i \int_{t_i}^{t_{i+1}} \rho_t^2 dt
\] (5.4)
is small enough. Then \(M_t^{S,s,x}\) converges, as \(\varepsilon(S)\) tends to 0, to the martingale \(M_t^{s,x}\) with limit \(\phi(X_0^{s,x})\) and \(f(s,x)\) is defined to be \(M_t^{s,x}\); it follows that \(f\) is measurable and \(f(0,) = \phi\). Moreover, using the Markov property,
\[
M_t^{S,s,x} = M_t^{S,t,X_0^{s,x}}
\] (5.5)
so, by taking the limit,
\[
M_t^{s,x} = f(t, X_t^{s,x}).
\] (5.6)
Thus \(f(t, X_t^{s,x})\) is a martingale with bounded quadratic variation. \(\square\)

In the previous theorem \(f\) is only measurable; however a better regularity result can be checked from Theorem 2.3.1.

**Proposition 5.2.** Under the assumptions of Theorem 5.1, suppose moreover that
\[
\sup_t \mathbb{E} \sup_x \left| \phi'(X_0^{t,x}) \frac{\partial X_0^{t,x}}{\partial x} \right| < \infty.
\] (5.7)
Then the function \(f\) is Lipschitz with respect to \(x\) uniformly in \((t, x)\) and is continuous with respect to \((t, x)\).

**Proof.** Fix a time \(t\) and two points \(x_0, x_1\) of \(U\). Let \(x(\alpha), 0 \leq \alpha \leq 1\) be a minimizing geodesic in \(U\) between these two points (for the Riemannian metric induced by the Euclidean metric). Then \(\phi(X_0^{t,x(\alpha)})\) is smooth with respect to \(\alpha\) and from (5.7), we have
\[
\mathbb{E} \sup_{\alpha} \left| \frac{\partial \phi(X_0^{t,x(\alpha)})}{\partial \alpha} \right| \leq C \delta_U(x_0, x_1).
\] (5.8)
We deduce that \(f\) is uniformly Lipschitz with respect to \(x\) by applying Theorem 2.3.1. We still have to prove that \(f\) is continuous with respect to \(t\) at any \((t_0, x_0)\); using the
Lipschitz continuity which has already been proved, the set \( C(t_0) \) of points \( x \) such that \( f \) is continuous at \((t_0, x)\) is closed; for any \( s < t_0 \), the process \( f(t, X^{s,x}_t) \) is almost surely continuous at \( t_0 \) (it is a martingale); the process \( X^{s,x}_t \) is also continuous. Now use
\[
\delta\left(f(t, X^{s,x}_t), f(t_0, X^{s,x}_t)\right) \leq \delta\left(f(t, X^{s,x}_t), f(t, X^{s,x}_{t_0})\right) + \delta\left(f(t, X^{s,x}_{t_0}), f(t_0, X^{s,x}_t)\right).
\] (5.9)

We deduce from the previous remarks that the two terms of the right-hand side converge to 0 as \( t \to t_0 \), so \( f(t, X^{s,x}_t) \) converges almost surely to \( f(t_0, X^{s,x}_t) \); thus \( C(t_0) \) contains the closed support of the law of \( X^{s,x}_{t_0} \); by letting \( s \uparrow t_0 \), we deduce that it contains \( x_0 \).

We can also give (as in [9] for regular geodesic balls) a probabilistic solution to the Dirichlet problem in a small convex domain. The framework which was set in the introduction (the exit problem) is a particular case of the following one.

**Theorem 5.3.** On a state space \( U \), consider a time-homogeneous Markov process which can be realized as a stochastic flow \( X^x_t \) defined on our Wiener space. Let \( \Delta \) be a domain satisfying the assumptions of Theorem 4.1 and let \( \phi(X^x) \) be a bounded \( \Delta \)-valued variable which is measurable with respect to the asymptotic \( \sigma \)-field of the Markov process. Then there exists a measurable function \( f \) from \( U \) to \( \Delta \) such that for any \( x \), \( f(X^x_t) \) is a martingale converging to \( \phi(X^x) \).

**Proof.** Theorem 4.1 enables the construction of a map \( \Psi \) defined on the set of \( \Delta \)-valued Wiener functionals \( L \) such that \( \Psi(L) \) is a martingale with final value \( L \); denote by \( \Psi_0(L) \) the value at time 0 of this martingale; if \( L \) is decomposed as
\[
L = L(W_s, s \leq t; W_s - W_t, s > t)
\] (5.10)
then one can check that the value at time \( t \) of this martingale is almost surely
\[
M_t = \Psi_0(L(W_s, s \leq t; .)).
\] (5.11)
In the framework of the theorem, define
\[
f(x) = \Psi_0(\phi \circ X^x).
\] (5.12)
Then $f$ is measurable and using the Markov property for the flow $X^x_t$, the martingale with final value $\phi(X^x)$ is from (5.11)

$$M^x_t = \Psi_0(\phi \circ X^x_t) = f(X^x_t).$$  

(5.13)

The only regularity on $f$ proclaimed in Theorem 5.3 is the measurability; analytical methods may provide better regularity results but we have to notice that our framework is more general; for the continuity, one can try to use coupling methods as in [9]. If $\Delta$ is not simply connected, we can use its universal cover as in §4; for instance, we can solve the Dirichlet problem for connected manifolds with non-positive sectional curvatures. Note also that the method of Proposition 4.3 can also be used to solve the Dirichlet problem; choose a $V$-valued function $g$ such that $g(X^x_t)$ converges to $\phi(X^x)$ as $t \to \infty$; then consider the family of martingales with limit $g(X^x_t)$ and let $t$ tend to infinity; one can try to prove that this procedure provides a solution in the homotopy class of $g$.

References


