NUMERICAL NULL CONTROLLABILITY OF THE HEAT EQUATION THROUGH A VARIATIONAL APPROACH

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Abstract. This work is concerned with the numerical computation of null controls for the heat equation. The goal is to compute a control that drives (an approximation of) the solution from a prescribed initial state at $t = 0$ to zero at $t = T$.

In spite of the diffusion of the heat equation, recent developments indicate that this issue is difficult and still largely open. Most of the existing literature, concerned with controls of minimal L^2-norm, make use of dual convex arguments and introduce backward adjoint system. In practice, the null control problem is then reduced to the minimization of a dual conjugate function with respect to the final condition of the adjoint state. As a consequence of the highly regularizing property of the heat kernel, this final condition - which may be seen as the Lagrange multiplier for the null controllability condition - does not belongs to L^2, but to a much larger space than can hardly be approximated by finite (discrete) dimensional basis. This phenomenon, unavoidable whatever be the numerical approximation used, strongly deteriorates the efficiency of minimization algorithms.

In this work, we do not use duality arguments and in particular do not introduce any backward heat equation. For the boundary case, the approach consists, first, in introducing a class of functions satisfying *a priori* the boundary conditions in space and time - in particular the null controllability condition at time T-, and then finding among this class one element satisfying the heat equation. This second step is done by minimizing a convex functional, among the admissible corrector functions of the heat equation. The inner case is performed in a similar way.

We present the (variational) approach, discuss the main features of it, and then describe some numerical experiments highlighting the interest of the method.

The method holds in any dimension but, for the sake of simplicity, we provide details in the one-space dimensional case.

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1. Introduction

We are concerned in this work with the null controllability problem for the 1D heat equation for both the boundary and the inner case. We denote by T any strictly positive real, ω any non-empty (small) subset of $(0,1)$ and 1_ω the characteristic function of ω. We introduce the diffusion function a assumed to be uniformly bounded and strictly positive all over the interval $(0,1)$:

$$a \in C^1([0,1]), \quad a(x) \geq a_0 > 0 \quad \forall x \in [0,1]. \tag{1.1}$$

We also introduce in the sequel the notation

$$q_T = \omega \times (0,T), \quad Q_T = (0,1) \times (0,T), \quad \Sigma_T \in \{1\} \times (0,T). \tag{1.2}$$

The boundary control problem we consider here can be stated as follows : given any initial data $u_0 \in L^2(0,1)$, find a control function $w \in L^2(\Sigma_T)$ such that the unique solution $u \in C^0([0,T]; H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1))$ of the homogeneous linear equation

$$\begin{cases} u_t - (a(x)u_x)_x = 0 & (x,t) \in Q_T, \\ u(x,0) = u_0(x) & x \in (0,1), \\ u(0,t) = 0, u(1,t) = w(t) & t \in (0,T) \end{cases} \tag{1.3}$$

satisfies the null controllability condition

$$u(\cdot,T) = 0 \quad \text{in} \quad (0,1). \tag{1.4}$$

Similarly, the inner (or distributed) control problem may be stated as follows: given any initial data $u_0 \in L^2(0,1)$, find a control function $f \in L^2(q_T)$ such that the unique solution $u \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1(0,1))$ of the homogeneous linear equation

$$\begin{cases} u_t - (a(x)u_x)_x = f\,1_\omega & (x,t) \in Q_T, \\ u(0,x) = u_0(x) & x \in (0,1), \\ u(t,0) = u(t,1) = 0 & t \in (0,T) \end{cases} \tag{1.5}$$

satisfies (1.4).

In the one dimensional space case, those controllability problems are known to be solvable since the seventies: we refer to the earlier contributions [9, 22] for some proofs based on spectral arguments. For more recent and general results based on duality arguments and Carleman type estimates, we refer to [10, 17]. As is usual in this type of problems, the dual approach allows to reduce the controllability problem to a suitable observability result for the adjoint system. Moreover, in the spirit of the celebrated Hilbert Uniqueness Method introduced by J.-L. Lions, they lead to a practical way of computing controls of a given minimal Sobolev norm.

In order to highlight the underlying difficulties that motivate the search of new methods, let us consider the inner case, which is simpler in many ways with respect to its boundary counterpart. Since there are controls $f \in L^2(q_T)$ for (1.5), it is natural to look for the one with minimal L^2-norm, that is, one seeks to minimize the quadratic functional $J(v) = \frac{1}{2}\|v\|^2_{L^2(q_T)}$ over the non-empty set

$$\mathcal{C}(u_0,T) = \{\, (u,f) : f \in L^2(q_T), u \text{ solves (1.5) and satisfies (1.4).} \}$$

Since it is difficult to construct pairs in $\mathcal{C}(u_0,T)$ (and *a fortiori* minimizing sequences !), one may used, following [3], duality arguments to replace the constrained minimization of J by the unconstrained minimization of its conjugate function $J^\star$ defined as

$$J^\star(\varphi_T) = \frac{1}{2}\iint_{q_T} |\varphi|^2\,dx\,dt + \int_0^1 u_0(x)\varphi(x,0)dx$$

over $\varphi_T \in \mathcal{H}$ (that will be made precise below), where φ is the adjoint backward state associated with (1.5) such that $\varphi(\cdot, T) = \varphi_T$. The existence of a positive constant $C = C(\omega, T)$ (the so-called *observability constant*) such that $C(\omega, T)\|\varphi(\cdot, 0)\|^2_{L^2(0,1)} \leq \|\varphi\|^2_{L^2(q_T)}$ for all $\varphi_T \in L^2(0, 1)$ implies that $J^\star$ is coercive on the Hilbert space $\mathcal{H}$ defined as the completion of $\mathcal{D}(0, 1)$ for the norm $\|\varphi\|_{L^2(q_T)}$. The control f of minimal $L^2(q_T)$-norm is then given by $f = \hat{\varphi}\, 1_\omega$ where $\hat{\varphi}$ is associated with the unique minimizer $\hat{\varphi}_T$ in $\mathcal{H}$ of $J^\star$ (see [23]). The difficulty, when one wants to approximate such control, that is when one likes to minimize numerically $J^\star$, is that the space $\mathcal{H}$ is huge, in particular, contains H^{-s} for every $s \in \mathbb{N}$, and even elements that may not be distributions. Numerical experiments do suggest that the minimizer $\hat{\varphi}_T$ is very singular (we refer to [3] and also to [20] for more details). Notice that this phenomenon is independent of the choice of J, but is related to the use of dual variables. As we stressed in the abstract, the equality (1.4) can be viewed as an equality in a very small space (due to the strong regularization effect of the heat kernel). Accordingly, the associated multiplier φ_T must belong to a large dual space, much larger than $L^2(0, 1)$, that cannot be represented numerically. We refer to [7], generalizing [3] for weighted-norms, where the same ill-posedness is shown and to [12] where a Tikhonov regularization is introduced and analyzed. For these reasons, robust numerical approximations of null controls for parabolic systems remain a challenge.

Recently, an alternative way of looking at these problems and avoiding the introduction of dual variables has been introduced in [21]. It is based on the following simple strategy. Instead of working all the time with solutions of the underlying state equation, and looking for one that may comply with the final desired state, one considers a suitable class of functions complying with required initial, boundary, and final conditions, and seeks one of those that is a solution of the state equation. This is in practice accomplished by setting up an error functional defined for all feasible functions, and measuring how far those are from being a solution of the underlying state equation. The task of showing that a problem is controllable amounts to proving that the infimum of the error is a minimum (there is a global minimizer of the error), and that it vanishes. This job requires some interesting analysis as the error functional is not a local, classical integral functional but rather a non-local functional as the ones consider for optimal control for distributed parameter systems ([18]). Once we have a feasible function with zero error, the control is obtained as the trace of this optimal function (or some other function determined in a unique way through it) in the set where we are entitled to act on the system.

One main practical advantage of this variational approach is that the way to get closer to a solution of the problem is by minimizing a functional that cannot get stuck on local minima because the only critical points of the error turn out to be global minimizers with zero error (see next section). Therefore a general strategy for numerical approximation consists in using a typical descent algorithm for this error functional. Exploring this possibility for the problems described above is the main reason for this paper. It is organized as follows. In Section 2, we describe (in a non-technical way) the main ingredients of the variational approach for the heat equation for the boundary case first, following [21]. We then show that [21] may be adapted to address the inner situation. We then move on to provide the details for the numerical approach based on the Polak-Ribière version of the conjugate gradient algorithm to minimize the error functional. Section 4 presents several experiments and discuss the practical interest of the approach. Section 5 treats a typical non-linear example to stress the flexibility of the approach. The final section provides a simple method to reduce the cost of controls.

To our knowledge, very few contribution on that topic has appeared since the seminal paper of Carthel-Glowinski-Lions [3] devoted to approximate controllability using duality. This is due to the intrisic ill-posedness of the problem we have just pointed out. For the null boundary case in one dimensional space, we mention the motion planning method introduced in [16] allowing a semi-explicit expression of controlled solutions in term of Gevrey series. This approach has been adapted and numerically developed recently in [20] to obtain inner controls. The recent works [7, 8] - following [10] - extend [3] with Carleman weighted L^2-norm. For a numerical analysis viewpoint, we also indicate contributions [2, 5, 15].

2. The variational approach of the null controllability

We are going to describe in this section the basic ingredients of the variational approach in order to apply it to both boundary, and inner controllability problems for the 1D heat equation.

2.1. **Boundary controllability.** Consider first the boundary controllability problem for the heat equation which consists in finding a function $w \in L^2(\Sigma_T)$, such that the solution of the problem (1.3) will comply with $u(x,T)=0$ in $(0,1)$, so that the state u with initial distribution given by initial data u_0 is led to state 0 at time T under the action of the boundary control w at the right-end point $x=1$. The data u_0 is given *a priori*, and the function a is assumed to be uniformly bounded and strictly positive all over the interval $(0,1)$.

The main idea of the variational method, as introduced in [21], consists in setting up an error functional which measures the deviation of functions from being a solution of the underlying heat equation, and minimizing such error over the class of feasible functions that comply with initial, boundary, and final conditions. Namely, consider the class of functions

$$\mathcal{A}=\left\{u \in H^1(Q_T) : u(x,0)=u_0(x), u(x,T)=0, x\in(0,1), u(0,t)=0, t\in(0,T)\right\}$$

assumed non empty. This requirement simply demands some compatibility with the vanishing boundary data for $x=0$, precisely that $u_0(0)=0$ and that u_0, as the trace of an H^1 function over Q_T, be slightly more regular than $L^2(0,1)$, that is $u_0\in H^{1/2}(0,1)$. According to the regularizing effect of the heat kernel, this assumption may be removed if we assume that the control is zero at time $t=0$. For any $u\in\mathcal{A}$, we define its corrector v over Q_T as the solution of the (elliptic) problem

$$\begin{cases} u_t - v_{tt} - (a(x)(u_x+v_x))_x = 0, & (x,t)\in Q_T,\\ v_t(x,0)=v_t(x,T)=0, & x\in(0,1),\\ v(0,t)=v(1,t)=0, & t\in(0,T). \end{cases} \tag{2.1}$$

Notice that the unique solution of this problem is the minimizer over $H^1_{0,x}(Q_T)=\{v\in H^1(Q_T), v=0 \,\text{on}\, \{0,1\}\times(0,T)\}$ of the regular quadratic functional

$$\frac{1}{2}\iint_{Q_T}\left((|v_t|^2+a(x)|v_x|^2)+u_tv+a(x)u_xv_x\right)dx\,dt.$$

The Neumann conditions on the part of the boundary for $t=0$, and $t=T$, are the natural boundary one. One may also consider Dirichlet conditions. Note how this variational problem determining the corrector v is a well-defined problem if $u\in\mathcal{A}$. Even though the corrector function v was introduced in [21] for each time slice t to preserve as general a framework as possible, from the point of view of numerical approximation it is advantageous to define such error function globally in the whole time-space domain Q_T by introducing the additional term $-v_{tt}$. This has a regularizing effect on the time dependence which is very convenient for numerics.

The error functional is then

$$E:\mathcal{A}\to\mathbb{R}^+,\quad E(u)=\frac{1}{2}\iint_{Q_T}\left(|v_t|^2+a(x)|v_x|^2\right)\,dx\,dt, \tag{2.2}$$

where v is the corrector associated with u. It turns out that our problem is controllable if and only if the minimum of the error vanishes. This amounts to proving two facts:

- the infimum of the error $m\ge 0$ is attained;
- it vanishes $m=0$.

The first part is achieved by using the direct method, and requires to deal with a quadratic, but possibly degenerate, functional. There is no special difficulty here except for showing that there are bounded minimizing sequences in spite of lack of coercivity. This was also a difficulty in [21] that was resolved by using the linear dependence of v on u, and the quadratic nature of the error functional, together with an easy lemma to overcome this lack of coercivity. The interesting point to stress is that this lack of coercivity means that not all minimizing sequences will be uniformly bounded, as they would under coercivity, but it suffices to show that there are some bounded minimizing sequences. More specifically, we have the following general fact, which was expressed in a different, and possibly less rigorous, way in [21].

Lemma 2.1. *Let $E : X \to \mathbb{R}$ be a quadratic, convex, non-negative functional defined on a Hilbert space X. Let H be a closed subspace, and $u_0 \in X$. Suppose that*

$$E(u_0) \leq \liminf_{j\to\infty} E(u_0 + t_j u_j) \tag{2.3}$$

whenever $t_j \to \infty$ and $u_j \rightharpoonup 0$. Then E attains its infimum over the manifold $u_0 + H$.

Proof. Without loss of generality, we can assume that $\inf_{u_0+H} E = 0$, for otherwise we replace E by $E - m$ if $m = \inf_{u_0+H} E > 0$. Let $\{u_j\} \subset H$ be such that $E(u_0 + u_j) \searrow 0$, and that $\|u_j\| \to \infty$. The normalized sequence $\{u_j/\|u_j\|\}$ is uniformly bounded in X, and hence, at least for a subsequence which we do not care to relabel, it converges weakly in X to some $\tilde{u}$. If $\tilde{u} = 0$, then our explicit hypothesis on the statement ensures that u_0 is a minimizer of E. Suppose it is not. By Mazur's lemma, we have that

$$\tilde{U}_j = \sum_{i=m_j}^{n_j} \bar{t}_i^{(j)} \frac{u_i}{\|u_i\|} \to \tilde{u} \text{ strong in } X,$$

for certains weights $\bar{t}_i^{(j)}$ with $m_j \to \infty$ when $j \to \infty$. Put

$$t_i^{(j)} = \frac{\bar{t}_i^{(j)}}{\|u_i\| \sum_k \frac{\bar{t}_k^{(j)}}{\|u_k\|}},$$

and

$$U_j = \frac{\sum_{i=m_j}^{n_j} t_i^{(j)} u_i}{\| \sum_{i=m_j}^{n_j} t_i^{(j)} u_i\|} \to u \text{ strong in } X,$$

where u is unitary, because U_j is just a multiple of $\tilde{U}_j$. By the convexity of E, we can write

$$E\left(u_0 + \|\sum_{i=m_j}^{n_j} t_i^{(j)} u_i\| U_j\right) = E\left(u_0 + \sum_{i=m_j}^{n_j} t_i^{(j)} u_i\right) \leq \sum_{i=m_j}^{n_j} t_i^{(j)} E\left(u_0 + u_i\right) \searrow 0. \tag{2.4}$$

Consider the parabolas $g_j(t) = E(u_0 + tU_j)$, and let t_j be a point of minimum for each j, so that $g_j(t_j) \searrow 0$ because $g_j(t_j)$, by definition, is less than or equal to the left-hand side in (2.4). If the sequence $\{t_j\}$ (or some subsequence) can be chosen bounded, then we have a uniformly bounded minimizing sequence. If $t_j \to \infty$, since t_j depends continuously on U_j, and $U_j \to u$ strong, we conclude that necessarily $E(u) = 0$, because the infimum of the parabola $g(t) = E(u_0 + tu)$ would be achieved for t going to infinity, and, since E is quadratic and non-negative, that can only happen if g is constant at $E(u_0)$, and $E(u) = 0$. In this case, if $\pi^\perp$ is the projection onto the orthogonal complement of $\{u\}$ ($u \neq 0$) in H, the sequence $\{u_0 + \alpha_j \pi U_j\}$ is also minimizing (precisely because $E(u) = 0$)

$$E\left(u_0 + \alpha_j \pi U_j\right) = E\left(u_0 + \alpha_j \pi U_j + \alpha_j \langle U_j, u\rangle u\right) = E\left(u_0 + \alpha_j U_j\right) \searrow 0,$$

but has no component in the u direction. If $\{\alpha_j \pi U_j\}$ is not uniformly bounded in X, we repeat this whole process starting out with the minimizing sequence $u_j = \alpha_j \pi U_j$ in the proper subspace πH.

We would finish with a uniformly bounded minimizing sequence. Because of the convexity of E, it attains its infimum somewhere in $u_0 + H$. □

It is interesting to point out that condition (2.3) is essential. We have the following counterexample, without this condition. Let $X = \ell^2$, and for $a = (a_j) \in \ell^2$, put

$$E(a) = \sum_{j=1}^{\infty} 2^{-j}(a_j - j)^2.$$

It is clear that the infimum vanishes, but it is not taken on in ℓ^2. Condition (2.3) fails here because in ℓ^2 weak convergence does not necessarily mean oscillatory behavior. In particular, normalized minimizing sequences do converge weakly to zero.

The application of this lemma in our setting is straightforward, so that the error in (2.2) attains its minimum over $\mathcal{A}$, provided we check condition (2.3). To this aim, suppose that $u^{(0)} \in \mathcal{A}$, and let $u \in \mathcal{A}_0$ where

$$\mathcal{A}_0 = \left\{ U \in H^1(Q_T) : U(x,0) = U(x,T) = 0, x \in (0,1), U(0,t) = 0, t \in (0,T) \right\} \tag{2.5}$$

will be used later. Let $v^{(0)}$ and v be the correctors for $u^{(0)}$ and u, respectively, determined through (2.1). The error functional in (2.2) can also be expressed by setting

$$E(u^{(0)} + u) = \frac{1}{2} \iint_{Q_T} \left(|v_t + v_t^{(0)}|^2 + a(x)|v_x + v_x^{(0)}|^2 \right) dx\, dt.$$

Let us check condition (2.3) for this functional.

If $u^{(j)}$ is an arbitrary sequence in $\mathcal{A}_0$ converging weakly to zero, and $t_j \to \infty$, it is clear that

$$E(u^{(0)} + t_j u^{(j)}) = \frac{1}{2} \iint_{Q_T} \left(|t_j v_t^{(j)} + v_t^{(0)}|^2 + a(x)|t_j v_x^{(j)} + v_x^{(0)}|^2 \right) dx\, dt$$

if $v^{(j)}$ is the corrector associated with $u^{(j)}$ again through (2.1). Likewise

$$E(u^{(0)}) = \frac{1}{2} \iint_{Q_T} \left(|v_t^{(0)}|^2 + a(x)|v_x^{(0)}|^2 \right) dx\, dt.$$

We should show that indeed

$$\liminf_{j\to\infty} \iint_{Q_T} \left(|t_j v_t^{(j)} + v_t^{(0)}|^2 + a(x)|t_j v_x^{(j)} + v_x^{(0)}|^2 \right) dx\, dt \geq \iint_{Q_T} \left(|v_t^{(0)}|^2 + a(x)|v_x^{(0)}|^2 \right) dx\, dt.$$

Notice first that $u^{(j)} \rightharpoonup 0$ clearly implies $v^{(j)} \rightharpoonup 0$ as well. Thus,

$$\iint_{Q_T} v_t^{(j)} v_t^{(0)}\, dx\, dt, \iint_{Q_T} a(x) v_x^{(j)} v_x^{(0)}\, dx\, dt \to 0.$$

We can select $Q_j \subset Q_T$ so that $|Q_T \setminus Q_j| \searrow 0$ and

$$\iint_{Q_j} v_t^{(j)} v_t^{(0)}\, dx\, dt = \iint_{Q_j} a(x) v_x^{(j)} v_x^{(0)}\, dx\, dt = 0$$

for every j. Then

$$\begin{aligned}
\iint_{Q_T} |t_j v_t^{(j)} + v_t^{(0)}|^2\, dx\, dt &\geq \iint_{Q_j} |t_j v_t^{(j)} + v_t^{(0)}|^2\, dx\, dt \\
&= \iint_{Q_j} \left(t_j^2 |v_t^{(j)}|^2 + 2 t_j v_t^{(j)} v_t^{(0)} + |v_t^{(0)}|^2 \right) dx\, dt \\
&\geq \iint_{Q_j} |v_t^{(0)}|^2\, dx\, dt.
\end{aligned}$$

Likewise, we have

$$\iint_{Q_T} a(x)|t_j v_x^{(j)} + v_x^{(0)}|^2\, dx\, dt \geq \iint_{Q_j} a(x)|v_x^{(0)}|^2\, dx\, dt.$$

We conclude by taking limits in j.

Once we know that the infimum $m \geq 0$ is a minimum, we turn to optimality. We define, in a classical way, the variation of E in the direction $U \in \mathcal{A}_0$

$$< E'(u), U >= \lim_{t\to 0} \frac{E(u+tU) - E(u)}{t}.$$

where the set $\mathcal{A}_0$ of admissible variations of u is defined by (2.5). We easily obtain that

$$< E'(u), U >= \iint_{Q_T} (v_t V_t + a(x) v_x V_x)\, dx\, dt \tag{2.6}$$

where $V \in H^1_{0,x}(Q_T)$ is the corrector function associated with $U \in \mathcal{A}_0$, that is, the solution of

$$\begin{cases} U_t - V_{tt} - (a(x)(U_x + V_x))_x = 0, & (x,t) \in Q_T, \\ V_t(x,0) = V_t(x,T) = 0, & x \in (0,1), \\ V(0,1) = V(1,t) = 0, & t \in (0,T). \end{cases} \tag{2.7}$$

Multiplying the state equation (2.7) by v, integrating by parts, and taking into account the boundary conditions on v and U, we transform (2.6) into

$$< E'(u), U >= -\iint_{Q_T} (U_t v + a(x) U_x v_x)\ dx\, dt, \quad \forall U \in \mathcal{A}_0.$$

Now, let us assume that $u \in \mathcal{A}$ is a minimizer for E, so that $< E'(u), U >= 0$ for all $U \in \mathcal{A}_0$. This equality implies that v satisfy the backward heat equation

$$\begin{cases} -v_t - (a(x)v_x)_x = 0, & (x,t) \in Q_T, \\ a(1)v_x(t,1) = 0, & t \in (0,T), \end{cases}$$

in addition to the boundary conditions

$$\begin{cases} v_t(x,0) = v_t(x,T) = 0, & x \in (0,1), \\ v(0,t) = v(1,t) = 0, & t \in (0,T). \end{cases}$$

For any positive time $T > 0$, this implies, by the unique continuation property, that the corrector v of u is zero, that $m = E(u) = 0$, and the corresponding minimizer u satisfies an homogeneous heat equation. Since u belongs to $\mathcal{A}$, the minimizer of E is then a controlled solution of the heat equation. As already said, the Dirichlet control we are looking for is simply obtained by taking the trace of u along Σ_T. As the trace on Σ_T of $u \in H^1(Q_T)$, the control obtained then belongs to $H^{1/2}(\Sigma_T) \subset L^2(\Sigma_T)$. Neumann controls may be obtained in a similar way. Notice that this argument implies that critical points can only occur at zero error, as already remarked in the Introduction.

We insist on the fact that this perspective relies on the minimization of the error functional, and does not make use of duality argument nor introduce any dual variable. For each u, the corrector v is the solution of an elliptic linear and well-posed problem in $H^1(Q_T)$. We refer to [7] where a different variational approach leading to an elliptic problem defined on Q_T has been introduced and analyzed.

Notice that, even though there might not be rigorous results to be applied for some particular situation, the decrease of the error to zero is a sure indication that the problem is being controlled.

Remark 2.2. *There are many ways to define the corrector v. One may, for instance, replace the state equation of (2.1) by the following equation*

$$u_t - v_{tt} - (a(x)u_x + v_x)_x = 0, \qquad (x,t) \in Q_T$$

leading to $E(u) = \frac{1}{2}\iint_{Q_T}(|v_t|^2 + |v_x|^2)\, dx\, dt$, and the same expression of the first derivative. The choice we made in (2.1) seems the closest to the notion of a corrector for the heat equation.

2.2. **Inner controllability.** Let us now turn to the inner controllability case. This time we assume that the control is acting on a small subset ω (for simplicity assumed independent of the time variable) of $(0,1)$. A first constructive approach applying the previous section could be as follows:

- First, compute the boundary controls acting on $\partial\omega \times (0,T)$ and driving the solution of the heat equation to rest on the part $((0,1) \setminus \omega) \times \{T\}$;

- then, compute the inner control acting on the whole domain q_T by driving to rest the solution on the part $\omega\times\{T\}$, taking as boundary Dirichlet condition on ∂q_T the boundary controls of the first step. The inner control acting on q_T is then a null control for the whole solution at time T.

Let us give and use a different and more direct approach following the ideas of the previous section. We put

$$\mathcal{A}=\left\{u\in H^1(Q_T): u(x,0)=u_0(x), u(x,T)=0, x\in(0,1), u(0,t)=u(1,t)=0, t\in(0,T)\right\}$$

and $\mathcal{A}_0=H^1_{0,x,t}(Q_T)$, that is to say, simply $\mathcal{A}_0=H^1_0(Q_T)$. In order to ensure that the solution u satisfies the homogeneous heat equation off $q_T=\omega\times(0,T)$, we consider the following error functional

$$E(u)=\frac{1}{2}\iint_{Q_T\setminus q_T}\left(|v_t|^2+a(x)|v_x|^2\right)\,dx\,dt \tag{2.8}$$

where the corrector v is defined in two pieces :

(1) off q_T:

$$\begin{cases} u_t-v_{tt}-(a(x)(u_x+v_x))_x=0, & (x,t)\in Q_T\setminus q_T,\\ v=0, & (x,t)\in\partial((0,1)\setminus\omega)\times(0,T),\\ v_t(x,0)=v_t(x,T)=0, & x\in(0,1)\setminus\overline{\omega}. \end{cases} \tag{2.9}$$

(2) in q_T:

$$\begin{cases} u_t-v_{tt}-(a(x)(u_x+v_x))_x=0, & (x,t)\in q_T,\\ v=0, & (x,t)\in\partial\omega\times(0,T),\\ v_t(x,0)=v_t(x,T)=0, & x\in(0,1)\cap\overline{\omega}. \end{cases} \tag{2.10}$$

Notice that the error associated with a feasible u only depends on the corrector v off q_T. However, we would like to be able to update u also inside q_T so as to lead (through the corrector) the error to zero. Then, proceeding as before and using that the corrector v is zero on ∂q_T, we obtain that the first variation of E in any direction $U\in\mathcal{A}_0$ is given by

$$<E'(u),U>=-\iint_{Q_T\setminus q_T}(U_tv+a(x)U_xv_x)\,dx\,dt,\quad\forall U\in\mathcal{A}_0.$$

Notice that the boundary terms on ∂q_T, due to the integration by parts in time, vanish thanks to the assumption $v=0$ on ∂q_T in (2.9). Then, writing that u is critical point of the error, we are left with

$$\begin{cases} -(1_{Q_T\setminus q_T}v)_t-(1_{Q_T\setminus q_T}a(x)v_x)_x=0, & (x,t)\in Q_T,\\ v=0, & (x,t)\times\partial((0,1)\setminus\omega)\times(0,T). \end{cases} \tag{2.11}$$

Together with the Neumann boundary condition $v_t(\cdot,0)=v_t(\cdot,T)=0$, (2.11) implies, for any $T>0$ that $v\equiv 0$ in $Q_T\setminus q_T$, and therefore $E(u)=0$. Finally, such critical point $u\in\mathcal{A}$ satisfies the following equation

$$u_t-((a(x)u_x)_x=(v_{tt}+(a(x)v_x)_x)\,1_{q_T},\quad (x,t)\in Q_T.$$

Therefore, a control acting on q_T constructed in this way is given by $v_{tt}+(a(x)v_x)_x\,1_\omega$, v being the corrector associated with the critical point u inside q_T. This control belongs at least to $H^{-1}(q_T)$.

Remark 2.3. *Since the controllability problem is formulated in Q_T, we may consider, without further change, the case where the support of the control depends on the time variables, i.e. $q_T=\{(x,t)\in Q_T: g(t)<x<h(t), t\in(0,T)\}$ where g and h are two smooth functions on $[0,T]$ with $0<g\le h<1$, $g(t)\neq h(t)$. We refer to [7] for some experiments using a different variational approach.*

3. Numerical resolution of the minimization problem

As shown in the previous section, the practical search of controls for the heat equation may be reduced to minimization for corrector problems. We describe in this section the minimization procedure to approximate numerically correctors. We give the details for the boundary case, and then point out the main differences for the inner counterpart.

For the boundary case, we have to solve

$$\begin{cases} \text{Minimize} \quad E(u) = \dfrac{1}{2} \displaystyle\iint_{Q_T} (|v_t|^2 + a(x)|v_x|^2)\, dx\, dt, \\ \text{subject to} \quad u \in \mathcal{A} \end{cases} \tag{3.1}$$

with $\mathcal{A} = \big\{u \in H^1(Q_T); u(x,0) = u_0 \text{ on } (0,1), u(\cdot,T) = 0 \text{ on } (0,1), u(x,t) = 0 \text{ on } \Sigma_T\big\}$. We endowed the space $\mathcal{A}$ with the scalar product

$$(u,v)_{\mathcal{A}} = \iint_{Q_T} (u_t\, v_t + a(x) u_x\, v_x + u\, v) dx dt, \quad \forall u, v \in \mathcal{A}$$

and note that $\|u\|_{\mathcal{A}} = \sqrt{(u,u)_{\mathcal{A}}}$ for all $u \in \mathcal{A}$. $\mathcal{A}_0$ is endowed with the same scalar product.

3.1. Conjugate gradient algorithm.

The Polak-Ribière version of the conjugate gradient (CG) algorithm to minimize E over $\mathcal{A}$ is as follows (see [13]):

- *Step 0: Initialization* Given any $\varepsilon > 0$ and any $u^0 \in \mathcal{A}$, compute the residual $g^0 \in \mathcal{A}_0$ solution of
$$(g^0, U)_{\mathcal{A}} =< E'(u^0), U > \quad \forall U \in \mathcal{A}_0.$$
If $\|g^0\|/\|u^0\| \leq \varepsilon$ take $u = u^0$ as an approximation of a minimum of E. Otherwise, set $z^0 = g^0$.
For $n \geq 0$, assuming u^n, g^n, z^n being known with g^n and z^n both different from zero, compute u^{n+1}, g^{n+1}, and if necessary z^{n+1} as follows:
- *Step 1: Steepest descent* Set $u^{n+1} = u^n - \lambda_n z^n$ where $\lambda_n \in \mathbb{R}$ is the solution of the one-dimensional minimization problem
$$\text{minimize} \quad E(u^n - \lambda z^n), \quad \text{over}\, \lambda \in \mathbb{R}. \tag{3.2}$$
Then, compute the residual $g^{n+1} \in \mathcal{A}_0$ from the relation
$$(g^{n+1}, U)_{\mathcal{A}} =< E'(u^{n+1}), U > \quad \forall U \in \mathcal{A}_0.$$
- *Step 2: Convergence testing and construction of the new descent direction.* If $\|g^{n+1}\|_{\mathcal{A}}/\|g^0\|_{\mathcal{A}} \leq \varepsilon$ take $u = u^{n+1}$; otherwise compute
$$\gamma_n = \frac{(g^{n+1}, g^{n+1} - g^n)_{\mathcal{A}}}{(g^n, g^n)_{\mathcal{A}}}, \qquad z^{n+1} = g^{n+1} + \gamma_n z^n. \tag{3.3}$$
Then do $n = n+1$, and return to step 1.

Let us provide more details for two important steps of the algorithm :

- Since E is a quadratic functional with respect to u, one may explicitly solve the problem (3.2): we write
$$E(u^n - \lambda z^n) = E(u^n) - \lambda \iint_{Q_T} (v_t^n Z_t^n + a(x) v_x^n Z_x^n)\, dx\, dt + \frac{\lambda^2}{2} \iint_{Q_T} (|Z_t^n|^2 + a(x)|Z_x^n|^2) dx dt$$
where Z_x^n is the corrector of z^n, solution of
$$\begin{cases} z_t^n - Z_{tt}^n - (a(x)(z_x^n + Z_x^n))_x = 0, \quad (x,t) \in Q_T, \\ Z(\cdot, t) = 0, \quad \Sigma_t, \quad Z_t(x, \cdot) = 0, \quad \Sigma_x \end{cases}$$

so that the optimal parameter is given by

$$\lambda_n = \frac{\iint_{Q_T} (v_t^n Z_t^n + a(x) v_x^n Z_x^n)\, dx\, dt}{\iint_{Q_T} (|Z_t^n|^2 + a(x)|Z_x^n|^2)\, dx\, dt} = -\frac{\iint_{Q_T} (z_t^n v^n + a(x) z_x^n v_x^n)\, dxdt}{\iint_{Q_T} (|Z_t^n|^2 + a(x)|Z_x^n|^2)\, dx\, dt}.$$

- The computation of the residual g^n is performed as follows. According to the equality

$$< E'(u^n), U >= -\iint_{Q_T} (U_t v^n + a(x) U_x v_x^n)\, dx\, dt, \quad \forall U \in \mathcal{A}_0,$$

$E'(u^n) \in H^{-1}(Q_T))$ may be identified with the linear functional on $\mathcal{A}_0$ defined by

$$U \rightarrow -\iint_{Q_T} (U_t v^n + a(x) U_x v_x^n)\, dx\, dt.$$

It then follows that g^n is the solution of the following linear variational problem : find $g^n \in \mathcal{A}_0$ such that

$$\iint_{Q_T} (g_t^n U_t + a(x) g_x^n U_x + g^n U)\, dx\, dt = -\iint_{Q_T} (U_t v^n + a(x) U_x v_x^n)\, dx\, dt, \quad \forall U \in \mathcal{A}_0,$$

where $v^n \in H^1_{0,x}(Q_T)$ is the corrector associated with u^n. The well-posed elliptic boundary value problem corresponding to this variational formulation is

$$\begin{cases} -g_{tt}^n - (a(x) g_x^n)_x + g^n = v_t^n + (a(x) v_x^n)_x & (x,t) \in Q_T \\ g^n(0,t) = 0,\ g_x^n(1,t) + v_x^n(1,t) = 0, & t \in (0,T) \\ g^n(x,0) = g^n(x,T) = 0, & x \in (0,1). \end{cases} \tag{3.4}$$

Remark 3.1. *As we mentioned above, the parameter γ_n given by (3.3) corresponds to the Polak-Ribière version of the conjugate gradient algorithm. In the present quadratic-linear situation, this one should coincide with the Fletcher-Reeves conjugate algorithm for which*

$$\gamma_n = \|g^{n+1}\|^2_{\mathcal{A}} / \|g^n\|^2_{\mathcal{A}}\,,$$

since gradients are conjugate to each other ($(g^m, g^n)_{\mathcal{A}} = 0$ for all $m \neq n$). However, we observed that in the parabolic situation (see also [8]*) the Polak-Ribière version (mainly used in nonlinear situations) allows to reduce the loss of the orthogonality, due to the numerical approximation.*

The detailed conjugate gradient scheme, written in a variational form, used for the minimization of E is then as follows :

Step 0: **Initialization** $u^0 \in \mathcal{A}$ be given, compute the corrector $v^0 \in H^1_{0,x}(Q_T)$ of u^0 solution of

$$\iint_{Q_T} (v_t^0 \phi_t + a(x) v_x^0 \phi_x)\, dx\, dt = -\iint_{Q_T} (u_t^0 \phi + a(x) u_x^0 \phi_x)\, dx\, dt, \quad \forall \phi \in H^1_{0,x}(Q_T),$$

then compute the gradient $g^0 \in \mathcal{A}_0$ solution of

$$\iint_{Q_T} (g_t^0 \phi_t + a(x) g_x^0 \phi_x + g^0 \phi)\, dx\, dt = -\iint_{Q_T} (v^0 \phi_t + a(x) v_x^0 \phi_x)\, dx\, dt, \quad \forall \phi \in \mathcal{A}_0,$$

and set $z^0 = g^0$.

Then, for $n \geq 0$, assuming u^n, g^n, z^n, v^n known, compute $u^{n+1}, g^{n+1}, z^{n+1}$ and v^{n+1} by :

Step 1: **Steepest descent** Compute the corrector $Z^n \in H^1_{0,x}(Q_T)$ of z^n solution

$$\iint_{Q_T} (Z_t^n \phi_t + a(x) Z_x^n \phi_x)\, dx\, dt = -\iint_{Q_T} (z_t^n \phi + a(x) z_x^n \phi_x)\, dx\, dt, \quad \forall \phi \in H^1_{0,x}(Q_T),$$

and set $u^{n+1} = u^n - \lambda_n z^n \in \mathcal{A}$ with

$$\lambda_n = -\frac{\iint_{Q_T} (z_t^n v^n + a(x) z_x^n v_x^n)\, dxdt}{\iint_{Q_T} (|Z_t^n|^2 + a(x)|Z_x^n|^2)\, dx\, dt}.$$

Next, compute the corrector $v^{n+1}\in\mathcal{A}_1$ of u^{n+1} solution of

$$\iint_{Q_T}(v_t^{n+1}\phi_t+a(x)v_x^{n+1}\phi_x)\,dx\,dt=-\iint_{Q_T}(u_t^{n+1}\phi+a(x)u_x^{n+1}\phi_x)\,dx\,dt,\quad \forall\phi\in H^1_{0,x}(Q_T),$$

and the gradient $g^{n+1}\in\mathcal{A}_0$ solution of

$$\iint_{Q_T}(g_t^{n+1}\phi_t+a(x)g_x^{n+1}\phi_x+g^{n+1}\phi)\,dxdt=-\iint_{Q_T}(v^{n+1}\phi_t+a(x)v_x^{n+1}\phi_x)\,dx\,dt,\quad \forall\phi\in\mathcal{A}_0.$$

Step 2: **Construction of the new descent direction**. If $\|g^{n+1}\|_{\mathcal{A}}/\|g^0\|_{\mathcal{A}}\le\varepsilon$, take $u=u^{n+1}$; otherwise compute

$$\gamma_n=\frac{(g^{n+1},g^{n+1}-g^n)_{\mathcal{A}}}{(g^n,g^n)_{\mathcal{A}}},\quad z^{n+1}=g^{n+1}+\gamma_n z^n.$$

Then do $n=n+1$, and return to step 1.

Once the convergence of the algorithm is reached, up to the threshold ε, we take the trace of u on Σ_T to define an approximation of the control w of (1.3): $w(t)=u(1,t)$, $t\in(0,T)$. We next compute an approximation of the controlled solution u by solving (1.3): the L^2-norm $\|u(1,T)\|_{L^2(0,1)}$, that may be seen as an *a posteriori* error, allows to evaluate the efficiency of the approach.

The minimization of the functional E related to the inner case (see (2.8)) is very similar. The main difference is that the corrector have to be solved independently in and off q_T (see (2.9) and (2)). The additional condition is $v=0$ on $\partial\omega\times(0,T)$. It is important to note that these correctors are linked through the descent direction g^n, solution of the problem posed in all of the domain Q_T :

(3.5)
$$\iint_{Q_T}(g_t^nU_t+a(x)g_x^nU_x+g^nU)\,dx\,dt=-\iint_{Q_T\setminus q_T}(U_tv^n+a(x)U_xv_x^n)\,dx\,dt,\quad \forall U\in\mathcal{A}_0=H^1_0(Q_T).$$

3.2. **Numerical approximation.** For "large" integers N_x and N_t , we set $\Delta x=1/N_x$, $\Delta t=T/N_t$, and $h=(\Delta x,\Delta t)$. Let us denote by $\mathcal{P}_{\Delta x}$ the uniform partition of $[0,1]$ associated with Δx, and let us denote by $\mathcal{Q}_h$ the uniform quadrangulation of Q_T associated with h. In particular,

$$\overline{Q}_T=\bigcup_{K\in\mathcal{Q}_h}K.$$

The following (conformal) finite element approximation of $H^1(Q_T)$ is introduced :

$$X_h=\{\varphi_h\in C^0([0,1]\times[0,T]):\varphi_h|_K\in(\mathbb{P}_{1,x}\otimes\mathbb{P}_{1,t})(K)\ \ \forall K\in\mathcal{Q}_h\}.$$

Here, $\mathbb{P}_{m,\xi}$ denotes the space of polynomial functions of order m in the variable ξ. Accordingly, the functions in X_h reduce on each quadrangle $K\in\mathcal{Q}_h$ to a polynomial of the form $A+Bx+Ct+Dxt$ involving 4 degrees of freedom. Obviously, the space X_h is a conformal approximation of $L^2(Q_T)$. We will also consider the space

$$X_{0h}=\{\varphi_h\in X_h:\varphi_h(0,t)=\varphi_h(1,t)=0\ \ \forall t\in(0,T)\},$$
$$X_{uh}=\{\varphi_h\in X_h:\varphi_h(0,t)=0\ \ \forall t\in(0,T),\varphi_h(x,0)=u_0(x),\varphi_h(x,T)=0\ \ \forall x\in(0,1)\}.$$

X_{uh} and X_{0h} are finite-dimensional subspace of $\mathcal{A}$ and $H^1_{0,x}(Q_T)$, respectively (and also of $L^2(0,T;H^1(0,1))$). The functions $\varphi_h\in X_{0h}$ are uniquely determined by their values at the nodes (x_j,t_j) of $\mathcal{Q}_h$ such that $0<x_j<1$.

Therefore, for any h, we consider the following problem, which is an approximation of (3.1):

$$\begin{cases}\text{Minimize}\quad E_h(u_h)=\dfrac{1}{2}\displaystyle\iint_{Q_T}(|v_{h,t}|^2+a(x)|v_{h,x}|^2)\,dx\,dt,\\ \text{subject to}\quad u_h\in X_{uh}.\end{cases}\tag{3.6}$$

According to the conjugate gradient algorithm, this minimization problem is reduced to the resolution of well-posed elliptic problems defined on Q_T in order to compute corrector functions $v_h \in X_{0h}$.

Once the optimal function u_h, minimizer of E over X_h is obtained, the control w_h is defined by $w_h = u_h$ on Σ_T. In order to check the quality of the control w_h, piecewise linear along Σ_T, one may compare such solution in X_{uh}, with the solution $\overline{u}_h$ of (1.3) starting from u_0 at time $t = 0$ and such that $\overline{u}_h = w_h$ on Σ_T. $\overline{u}_h$ is computed using, for the time discretization, the two-step implicit Gear scheme of order two in time (see for instance [14]). We set

$$\Phi_{\Delta x} = \{ \, z \in C^0([0,1]) : z|_k \in \mathbb{P}_{1,x}(k) \quad \forall k \in \mathcal{P}_{\Delta x} \, \},$$

a finite dimensional subspace of $L^2(0,1)$. Functions in $\Phi_{\Delta x}$ are uniquely determined by their values at the nodes of $\mathcal{P}_{\Delta x}$.

The Gear scheme, which is of order two, is then combined with a $\mathbb{P}_1$-finite element discretization in space as follows :

(1) We first set $u_h|_{t=0} = u_{0,\Delta x}$.

(2) Secondly, $u_h|_{t=t_1}$ is the solution of the linear problem in $\Psi \in \Phi_{\Delta x}$

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{\Delta t}(\Psi - \overline{u}_h|_{t=0}) z \, dx + \frac{1}{2} \int_0^1 a(x)(\Psi + \overline{u}_h|_{t=0})_x z_x \, dx = 0 \\ \forall z \in \Phi_{\Delta x} \, . \end{cases}$$

(3) For given $n = 2, \ldots, N_t - 1$, $\Psi^\star = \overline{u}_h|_{t=t_{n-1}}$ and $\overline{\Psi} = \overline{u}_h|_{t=t_n}$, $\overline{u}_h|_{t=t_{n+1}}$ is the solution of the linear problem in $\Psi \in \Phi_{\Delta x}$

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{2\Delta t}(3\Psi - 4\overline{\Psi} + \Psi^\star) z \, dx + \int_0^1 a(x) \Psi_x z_x \, dx = 0 \\ \forall z \in \Phi_{\Delta x} \, . \end{cases}$$

The L^2-norm $\|u_h(\cdot, T) - \overline{u}_h(\cdot, T)\|_{L^2(0,1)} = \|\overline{u}_h(\cdot, T)\|_{L^2(0,1)}$ allows to analyze *a posteriori* how the constraint (1.4) is satisfied. Recall that $\overline{u}_h$, obtained by an integration in time, solves the heat equation.

This same numerical approximation is used for the inner case.

4. Numerical experiments

We now present some numerical experiments, and analyze the behavior of the computed controls with respect to the data, and h. We assume for simplicity that $\Delta x = \Delta t$, that is we consider only uniform meshes $\mathcal{Q}_h$.

4.1. **Experiment 1: Boundary Case.**

As in [7, 8, 20], we assume that the function u_0 to be controlled is the first mode of the Laplacian, that is

$$u_0(x) = \sin(\pi x), \quad x \in (0,1).$$

for which the diffusion of (1.3), without control, i.e. $v = 0$, is the lowest. Moreover, we assume that the diffusion function a is constant equal to $a(x) = a_0 = 1/4$ in $(0,1)$, and take a controllability time equal to $T = 1/2$. We take a value a_0 lower than one in order to have a better control of the diffusion. Without control, these data leads to $\|u(\cdot, T)\|_{L^2(0,1)} = \sqrt{1/2} e^{-\pi^2/8} \approx 0.205$ and therefore leads to *stiff* case in the context of null boundary controllability for the heat equation.

We take $\varepsilon = 10^{-5}$ as the value for the stopping criterion of the conjugate gradient algorithm. The algorithm is initialized with $u^0 \in \mathcal{A}$ defined by $u^0(x,t) = u_0(x)(1 - t/T)^2$, $(x,t) \in Q_T$.

Table 1 gives various norms of the solution $u_h \in \mathcal{A}$ with respect to h, and clearly suggests the convergence of the approximation. Figure 1 depicts the evolution of $E(u^n)$ and the residual $\|g_h^n\|_{\mathcal{A}}$ (in $\log_{10}$-scale) with respect to the iteration of the conjugate gradient corresponding to $\Delta x = \Delta t = 1/100$. The algorithm requires 2 013 iterations to fulfill $\|g_h^n\|_{\mathcal{A}} \le \varepsilon$. As is typical when the heat equation is involved, the slope of the residual decreases significantly after the first iteration. This phenomenon is also possibly due to the lack of coercivity of E. We check however that the functional $E(u^n)$ decreases with respect to the iteration and reaches a small value, here of the order $\mathcal{O}(10^{-6})$.

$\Delta x = \Delta t$	1/25	1/50	1/100	1/200
$\sharp$ CG iteration	846	2 132	2 014	2 834
$\Vert u_h\Vert_{L^2(Q_T)}$	4.78×10^{-1}	5.06×10^{-1}	4.81×10^{-1}	4.87×10^{-1}
$\Vert u_h\Vert_{H^1(Q_T)}$	6.024	6.658	5.920	6.021
$\Vert u_h\Vert_{L^2(\Sigma_T)}$	1.369	1.487	1.392	1.418
$\Vert\overline{u}(\cdot,T)\Vert_{L^2(0,1)}$	1.95×10^{-2}	9.65×10^{-3}	8.39×10^{-3}	6.04×10^{-3}
$\Vert u_h-\overline{u}_h\Vert_{L^2(Q_T)}$	1.45×10^{-2}	6.31×10^{-3}	2.01×10^{-3}	9.34×10^{-4}
$E(u_h)$	4.88×10^{-6}	8.37×10^{-7}	1.22×10^{-6}	8.29×10^{-7}

TABLE 1. $u_0(x)=\sin(\pi x)$, $T=1/2$, $a_0=1/4$, $\Delta x=\Delta t=1/100$ - $\varepsilon=10^{-5}$ Numerical results with respect to $h=(\Delta x,\Delta t)$.

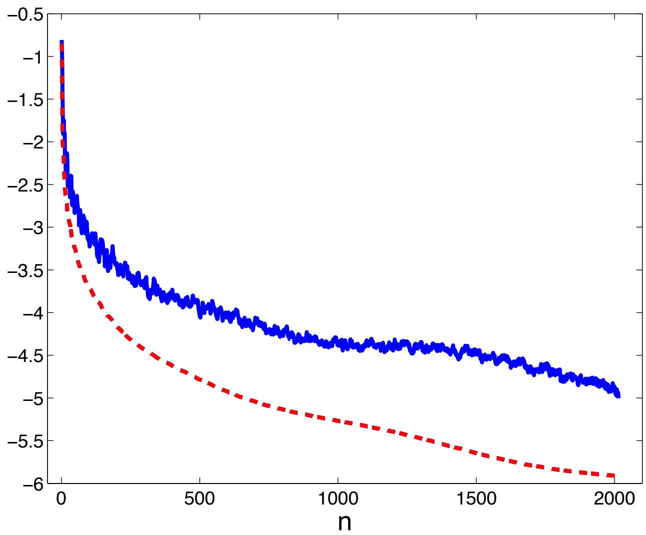

FIGURE 1. $u_0(x)=\sin(\pi x)$, $T=1/2$, $a_0=1/4$, $\Delta x=\Delta t=1/100$ - $\log_{10}$ of $E_h(u_h^n)$ (**dashed line**) and $\log_{10}(\Vert g_h^n\Vert_{\mathcal{A}})$ (**full line**) vs. the iteration n of the conjugate gradient algorithm.

Figure 2 depicts the corresponding solution $u_h\in\mathcal{A}$, and corrector $v_h\in H^1_{0,x}(Q_T)$. The trace of u_h on Σ_T is given in Figure 3-left. The control obtained is quite irregular near $t=T$. This is reminiscent of what it is obtained in [16] by computing exactly the controlled heat solution in the one dimensional space by means of the motion planning method (we also refer to [20] for an adaptation to the inner case using the so-called *transmutation method*). Finally, the solution $\overline{u}_h$ of (1.3) with $w(t)=u_h(1,t)$ is plotted at time T on Figure 3-right. We compute that the L^2-norm of $\overline{u}_h(1,T)$, what we called the *a posteriori* error, is $\Vert\overline{u}_h(\cdot,T)\Vert_{L^2(0,1)}\approx 8.39\times 10^{-3}$. This is an acceptable value that can be improved by reducing ε and h. Notice that the stiffness matrices involved in the resolution of the elliptic problems in step 1 are standard and well-conditioned. Notice also that a small gap between u and $\overline{u}$ (in particular at time T) is *a priori* unavoidable since they are approximated, and computed in a different way.

It is also interesting to note that this method allows to obtain non trivial controlled solution of the heat equation with zero initial data, that is in $\mathcal{A}_0$. Figure 4 depicts one such solution obtained with the initial function $u^0(x,t)=\sin(\pi x)t^2(1-t/T)^2$. For $\varepsilon=10^{-6}$, the algorithm converges after 1 242 iterations, and we get $E_h(u_h^{n=1\,242})\approx 6.63\times 10^{-9}$ and $\Vert\overline{u}_h(\cdot,T)\Vert_{L^2(0,1)}\approx 2.89\times 10^{-5}$. Accordingly, this means that any linear combination of such nontrivial solution in $\mathcal{A}_0$ with the

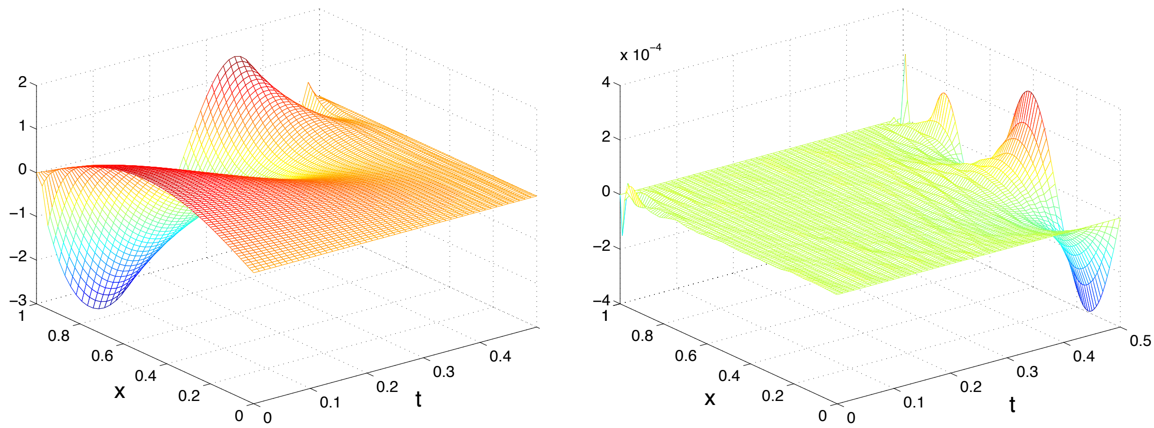

FIGURE 2. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Solution in $u_h \in \mathcal{A}_h$ (**Left**) and corresponding corrector v_h (**Right**) along Q_T.

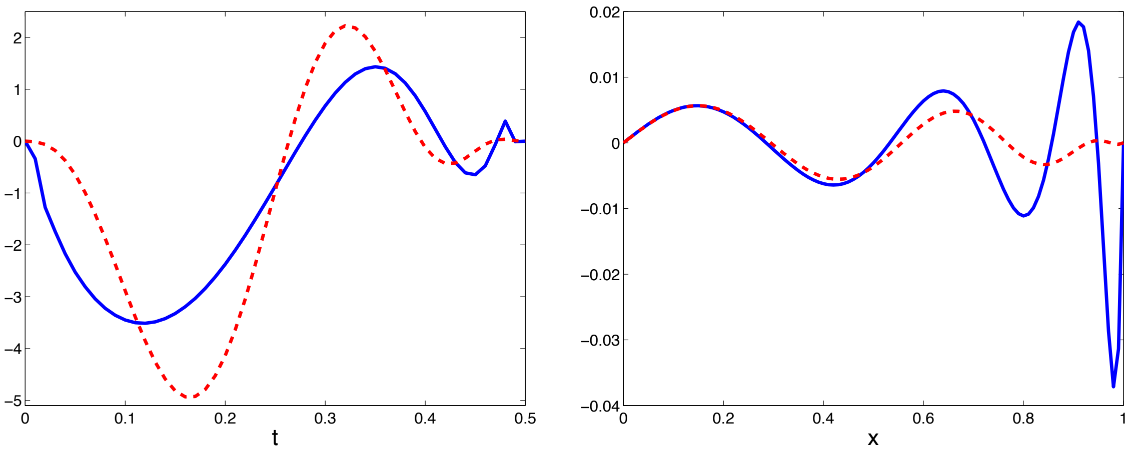

FIGURE 3. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - **Full line**: Trace $u_h(x = 1, t)$ vs. $t \in (0, T)$ (**Left**) and *a posteriori* solution $\overline{u}_h(x, T)$ vs. $x \in (0, 1)$ (**Right**); **Dashed line**: same quantities obtained with an additional compact support function in time.

previous ones in $\mathcal{A}$ remains a controlled solution of the heat equation. We will get back to this notion in Section 6. The non uniqueness of our minimization problem may also be checked by considering different initial function $u^0 \in \mathcal{A}$.

The experiments also confirm that the situation is more favorable, notably with respect to the speed of convergence of the algorithm, when the control acts on both sides, that is on $x = 0$ and $x = 1$. Figure 5 shows the controlled solution with initial data $u_0(x) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x)$ in that situation. For a same value of ε, the L^2-norm of the corrector as well as the *a posteriori* error are lower than in the previous situation: $E_h(u_h^{n=855}) \approx 6.69 \times 10^{-6}$ and $\|\overline{u}_h(\cdot, T)\|_{L^2(0,1)} \approx 2.21 \times 10^{-4}$ after 855 iterations.

We also emphasize that we may consider the more realistic situation where null Neumann boundary limit holds on the free part, here $x = 0$. It suffices to start with $u^0 \in \mathcal{A}_N = \{u \in H^1(Q_T), u(\cdot, T) = 0, u(\cdot, 0) = u_0, u_x(0, t) = 0\}$, and impose that both the descent direction and the corrector have null derivatives at $x = 0$. Figure 6 shows the function $u_h \in \mathcal{A}_N$ associated with $u_0(x) = \sin(\pi x)^2$. With $\varepsilon = 10^{-5}$, the convergence is reached after 3 431 iterations, and we

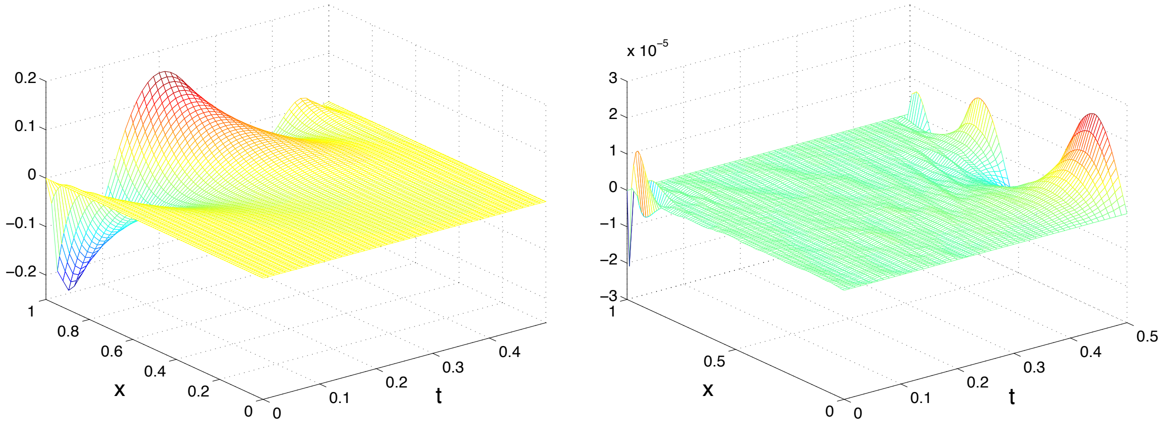

FIGURE 4. $u_0 = 0$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on 0 - $\varepsilon = 10^{-6}$- Solution in $u_h \in \mathcal{A}_h$ (**Left**) and corresponding corrector v_h (**Right**) along Q_T. $E_h(u_h^{n=1\,242}) \approx 6.63 \times 10^{-9}$ and $\|\overline{u}_h(\cdot, T)\|_{L^2(0,1)} \approx 2.89 \times 10^{-5}$.

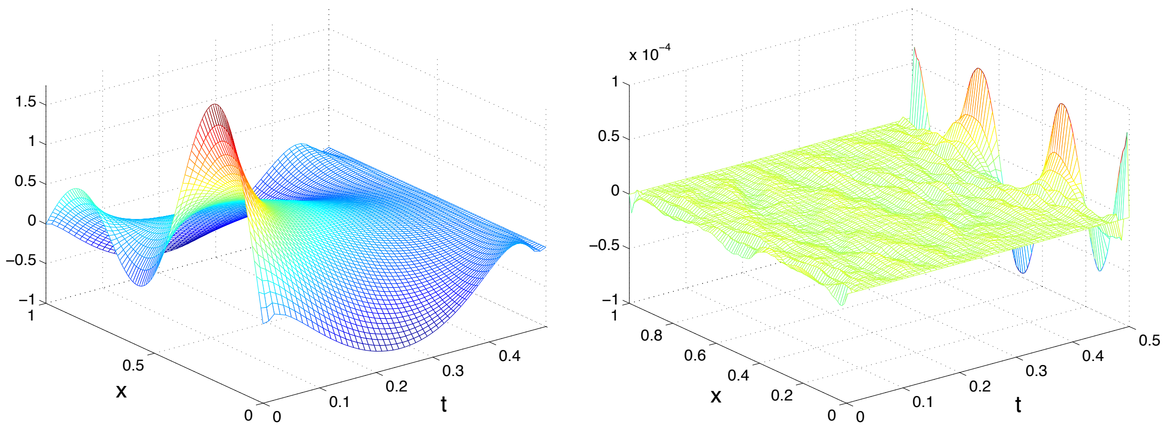

FIGURE 5. $u_0(x) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $\{0, 1\}$ - $\varepsilon = 10^{-5}$ - Solution in $u_h \in \mathcal{A}_h$ (**Left**) and corresponding corrector v_h (**Right**) along Q_T.

get $\|\overline{u}_h(\cdot, T)\|_{L^2(0,1)} \approx 1.31 \times 10^{-2}$. The convergence is slower in that case, because null Neumann boundary condition - contrary to null Dirichlet one - does not emphasize the dissipation of the solution.

Finally, let us comment on a simple way to smooth out the control near $t = T$, and therefore avoid the oscillations we mentioned at the beginning of this section (see Figure 3-Left). It suffices to replace at each iteration n the descent direction g^n by $c(t)g^n$ with any smooth positive function c such that $c(T) = c'(T) = 0$. Figure 3 gives (in dashed line) the quantities $u_h(1, \cdot)$ on $(0, T)$ and $\overline{u}(\cdot, T)$ on $(0, 1)$ obtained with $c(t) = \sin(\pi t/T)^2$ (in that case, notice that the solution is also smoothed at $t = 0$). This modification has the effect to reduce the *a posteriori* error $\|\overline{u}_h\|_{L^2(0,1)}$ but to increase the number of iterations. Notice also that the L^2-norm of the trace is larger (see Table 3).

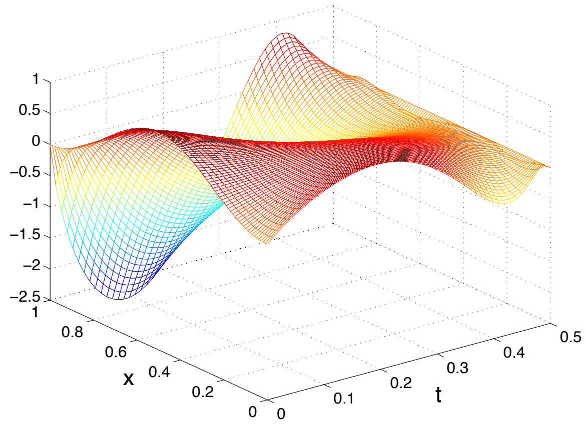

FIGURE 6. $u_0(x) = \sin(\pi x)^2$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Controlled solution u_h over Q_T with free Neumann boundary condition at $x = 0$.

$\Delta x = \Delta t$	$1/25$	$1/50$	$1/100$	$1/200$
$\sharp$ CG iteration	2 552	2 724	3 689	4 276
$\Vert u_h\Vert_{L^2(Q_T)}$	5.19×10^{-1}	5.26×10^{-1}	5.57×10^{-1}	5.71×10^{-1}
$\Vert u_h\Vert_{H^1(Q_T)}$	7.052	7.092	7.889	8.285
$\Vert u_h\Vert_{L^2(\Sigma_T)}$	1.526	1.554	1.678	1.738
$\Vert\overline{u}(\cdot,T)\Vert_{L^2(0,1)}$	9.08×10^{-3}	5.25×10^{-3}	3.46×10^{-3}	2.83×10^{-3}
$\Vert u_h-\overline{u}_h\Vert_{L^2(Q_T)}$	9.51×10^{-3}	2.73×10^{-3}	1.19×10^{-3}	9.61×10^{-4}
$E(u_h)$	2.88×10^{-6}	2.17×10^{-6}	1.20×10^{-6}	1.19×10^{-6}

TABLE 2. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - $\varepsilon = 10^{-5}$ Numerical results with respect to $h = (\Delta x, \Delta t)$ with a compact support function in time.

We observe similar results with Dirichlet boundary condition on $(0,1)\times\{0,T\}$ in (2.1). The CG algorithm converges faster and leads to a control with smaller L^2-norm. The *a posteriori* error $\Vert\overline{u}_h\Vert_{L^2(0,1)}$ is however larger.

4.2. **Experiment 2: Inner Case.** Let us consider the following data $\omega = (0.3, 0.6)$, $T = 1/2$ and $a(x) = a_0 = 1/4$ used notably in ([7, 8, 20]). The initial data to be controlled is again $u_0(x) = \sin(\pi x)$.

Table 1 collects some numerical values obtained with the CG algorithm and $\varepsilon = 10^{-6}$, $c(t) = \sin(\pi t/T)^2$. The situation is more favorable than the boundary case in the sense that the number of iterations to reach a relative residual of order 10^{-6} (instead of 10^{-5} in Table 1) is significantly reduced. As a consequence, the *a posteriori* error $\Vert\overline{u}_h(\cdot,T)\Vert_{L^2(0,1)}$ is smaller, of the order $\mathcal{O}(10^{-5})$. We also notice that the function $v_{h,tt} + (a(x)v_{h,x})_x$ is actually bounded with respect to h for the $L^2(Q_T)$-norm, and converges as $h \to 0$. This is the effect of the compactly support function $c(t)$, and an additional smoothing procedure on the descent direction g^n: g^n, solution of (3.5), is replaced by $\tilde{g}^n$, unique solution in $H^1_0(Q_T)$ of

$$(I - (\Delta x)^2\partial_{xx} - (\Delta t)^2\partial_{tt})\tilde{g}^n = g^n \ \text{ on } \ Q_T, \quad \tilde{g}^n = 0 \ \text{ on } \ \partial Q_T.$$

The controlled solution $u_h \in \mathcal{A}$ as well as the corresponding corrector v_h are depicted on Figure 7 for $(\Delta t, \Delta x) = (1/100, 1/100)$. The control function $v_{h,tt}+(a(x)v_{h,x})_x$, very small out of ω is given in Figure 8-left. This control, obtained with the initial guess $u^0(x) = \sin(\pi x)(1-t/T)^2$, is quite different from controls obtained by duality arguments in [20]. Mainly concentrated on the boundary of ω, his L^2-norm is larger : for $h = (1/100, 1/100)$, we obtain $\|v_{h,tt} + (a(x)v_{h,x})_x\|_{L^2(Q_T)} \approx 2.839$, about twice the HUM-control obtained in [20].

$\Delta x = \Delta t$	$1/25$	$1/50$	$1/100$	$1/200$
$\sharp$ CG iterates	135	192	231	361
$\|u_h\|_{L^2(Q_T)}$	2.53×10^{-1}	2.58×10^{-1}	2.57×10^{-1}	2.61×10^{-1}
$\|u_h\|_{H^1(Q_T)}$	1.301	1.336	1.337	1.352
$\|v_{tt} + (a(x)v_x)_x\|_{L^2(Q_T)}$	1.675	2.641	2.839	2.981
$\|\overline{u}_h(\cdot, T)\|_{L^2(0,1)}$	7.23×10^{-5}	5.43×10^{-5}	4.30×10^{-5}	2.91×10^{-5}
$\|u_h - \overline{u}_h\|_{L^2(Q_T)}$	3.21×10^{-5}	7.31×10^{-5}	5.10×10^{-5}	1.58×10^{-5}
$E(u_h)$	4.12×10^{-7}	3.34×10^{-7}	4.16×10^{-7}	2.36×10^{-7}

TABLE 3. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - $\omega = (0.3, 0.6)$ - $\varepsilon = 10^{-6}$ Numerical results with respect to $h = (\Delta x, \Delta t)$ with a compact support function in time.

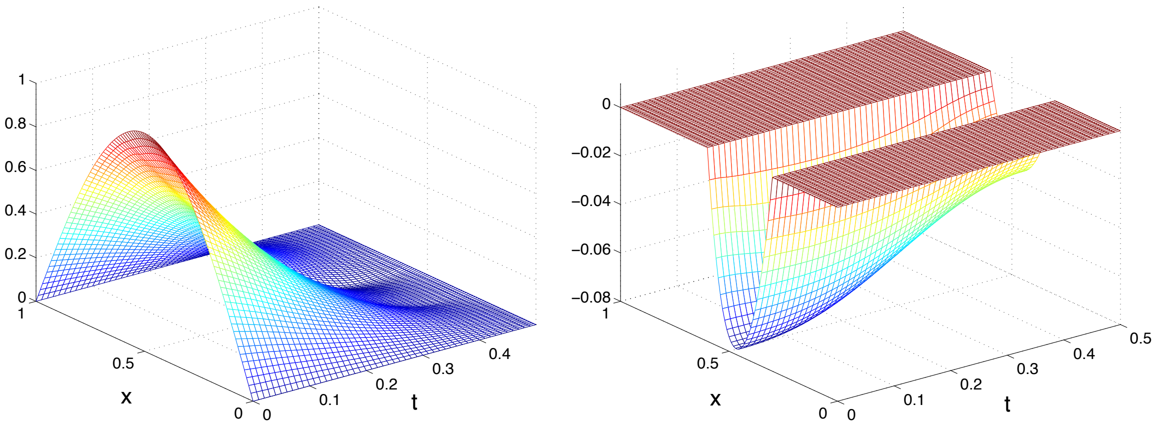

FIGURE 7. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $\omega = (0.3, 0.6)$ - $\varepsilon = 10^{-6}$ - Solution in $u_h \in \mathcal{A}_h$ (**Left**) and corresponding corrector v_h (**Right**) along Q_T.

5. REMARKS ON A NON-LINEAR SITUATION

As a good way to emphasize the flexibility of the variational approach to adapt itself to various different settings, we are going to indicate the changes needed for a typical non-linear situation where a low order non-linear perturbation is considered. Namely, we will look at the problem of finding a control w, so that the solution of the problem

$$\begin{cases} u_t - (a(x)u_x)_x + F(u) = 0, & (x,t) \in Q_T, \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,t) = 0, u(1,t) = w(t), & t \in (0,T) \end{cases} \tag{5.1}$$

will comply with $u(x,T) = 0$ for all $x \in (0,1)$. System (5.1) is known to be controllable, uniformly with respect to the data u_0 and T, if the nonlinear function $F(s)$ grows slower than $s \log^{3/2}(1 + |s|)$ as $|s| \to +\infty$ (we refer to [6]). Note that, to our knowledge, the numerical approximation of controls in that nonlinear context has not been addressed so far.

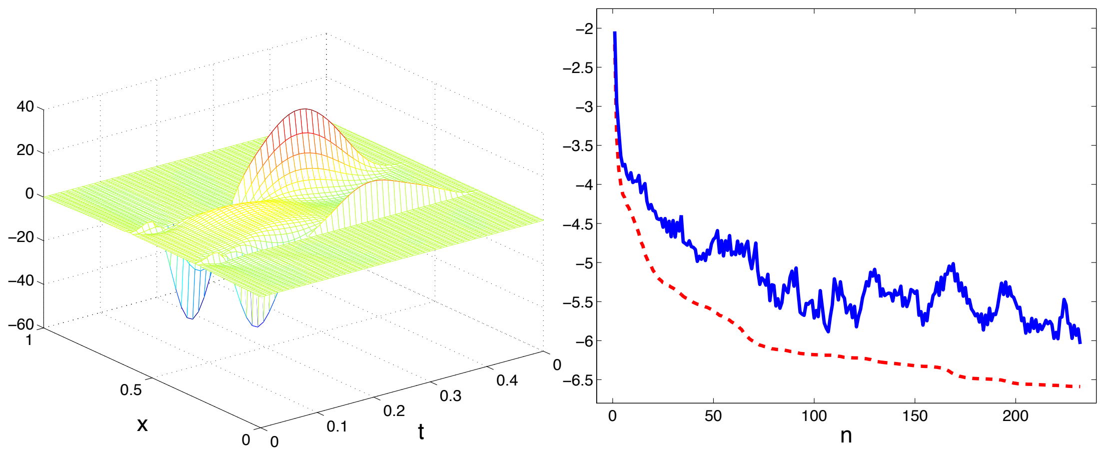

FIGURE 8. $u_0(x) = \sin(\pi x)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $\omega = (0.3, 0.6)$ - $\varepsilon = 10^{-6}$ - **Left**: Function $v_{tt} + (a(x)v_x)_x$ in Q_T; **Right**: $\log_{10}$ of $E_h(u_h^n)$ **(dashed line)** and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ **(full line)** vs. the iteration n of the CG algorithm.

The procedure for such nonlinear system is similar. We define the corrector associated with u, through the problem

$$\begin{cases} u_t - v_{tt} - (a(x)(u_x + v_x))_x + F(u) = 0, & (x,t) \in Q_T, \\ v_t(x,0) = v_t(x,T) = 0, & x \in (0,1), \\ v(0,t) = v(1,t) = 0, & t \in (0,T) \end{cases} \tag{5.2}$$

while the error functional E is still defined by (2.2). However, due to the nonlinear dependance of the corrector with respect to v, we cannot ensure, at least using the arguments we employed for the linear situation, that the infimum of E over $\mathcal{A}$ is reached. We then go on in a formal way assuming the well-posedness of the minimization problem. We obtain that the first derivative of E is given by

$$< E'(u), U >= - \iint_{Q_T} (U_t v + U_x v_x + (f'(u) \cdot U)v)\, dx\, dt, \quad \forall U \in \mathcal{A}_0,$$

leading to the characterization of the corrector v associated with any optimal u (assumed to exist in $\mathcal{A}$)

$$\begin{cases} v_t + (a(x)v_x)_x + F'(u)v = 0, & (x,t) \in Q_T, \\ v_t(x,0) = v_t(x,T) = 0, & x \in (0,1), \\ v(0,t) = v(1,t) = a(1)v_x(1,t) = 0, & t \in (0,T). \end{cases}$$

Once again, the solution of this system vanishes in Q_T so that the minimizer of E is a solution of the nonlinear heat equation (5.1).

As we mentioned earlier, even if we are not able to show the well-posedness of the minimization corrector problem, the decrease of the error to zero is a sure indication that the problem is being, at least approximately, controlled. Let us simply mention that, in the conjugate gradient algorithm, the function g^n in the steepest descent step is the solution of the linear formulation

$$\iint_{Q_T} (g_t^n \phi_t + a(x) g_x^n \phi_x) dx\, dt = - \iint_{Q_T} (v^n \phi_t + a(x) v_x^n + F'(u^n) v^n \phi) dx\, dt, \quad \forall \phi \in \mathcal{A}_0.$$

Figure 9 and 10 are concerned with the case $F(s) = -\alpha s \log(1 + |s|)$, $\alpha = 5$. F belongs to $C^1(\mathbb{R})$ and $F'(s) = -\alpha(\log(1 + |s|) + |s|/(1 + |s|))$. The other data are kept unchanged. This nonlinear term prevents the diffusion in time of the heat solution, that is the L^2-norm $\|u(\cdot,t)\|_{L^2(0,1)}$ increases with respect to t: in the uncontrolled situation, we get $\|u(\cdot,T)\|_{L^2(0,1)} \approx 1.240$ to be compared with $\|u(\cdot,T)\|_{L^2(0,1)} \approx 2.05\times 10^{-1}$ for the linear case. As for the linear situation, our approach allows to drive the solution in a closed neighborhood of zero: the *a posteriori* error we get is $\|\overline{u}_h(\cdot,T)\|_{L^2(0,1)} \approx 1.92\times 10^{-2}$. Compared to the linear situation, we observe also that the solution $u_h \in \mathcal{A}$ has a similar shape, the difference being the magnitude (see Figure 10-left). Notice that we have used the compact support function $c(t) = \sin^2(\pi t/T)$ so that u_h is smooth near T. The nonlinearity increases slightly the number of the iterations, here 2 788, to reach the same threshold $\varepsilon = 10^{-5}$. We also plot the difference $u_h - \overline{u}_h$ in Q_T (see Figure 10-right) so as to measure the influence of the corrector v_h. For larger values of α, the algorithm does not converge anymore. Similar phenomena are observed for smaller or larger values of T, for instance $T = 0.25$, and $T = 2$. We also obtain convergence results for the case $F(s) = -5|s|\log(1+|s|)$, more critical than the previous situation since f is non-positive. The number of iterations is greater (6 883) as well as the $L^2(\Sigma_T)$-norm of the control. Similar remarks hold for "more" nonlinear function such as $F(s) = \alpha|s|^p, p\in\mathbb{N}$ or $F(s) = \alpha\exp(s)$ provided that α or $\|u_0\|_{L^2(0,1)}$ be small enough.

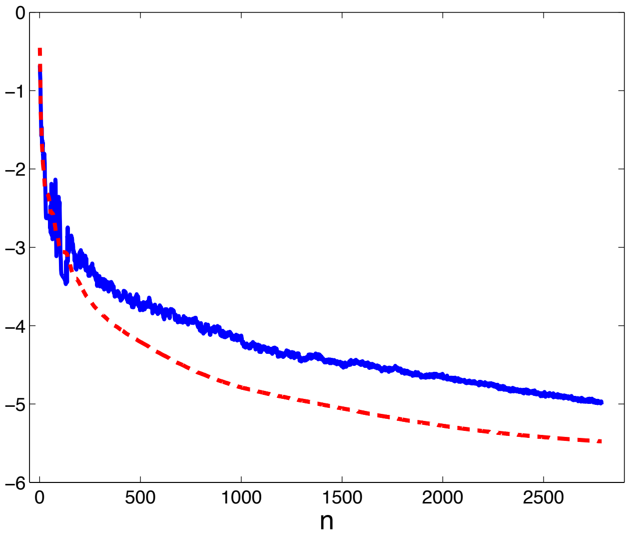

FIGURE 9. $u_0(x) = \sin(\pi x)$ - $F(s) = -5s\log(1+|s|)$ - $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - $\log_{10}(E_h(u_h^n))$ (**dashed line**) and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.

6. Reducing the norm of the control

By minimizing the error functional E defined by (2.2), we do not control any norm, in particular the L^2-norm, of the trace of the solution on Σ_T. From a practical viewpoint, it is interesting to minimize such norm. A possibility is to take advantage of the fact that the method allows to obtain non trivial controlled solutions in $\mathcal{A}$ with null initial condition u_0, that is, solutions in $\mathcal{A}_0$. Suppose a family $\{u_k\}_{k\in[1,N]}$ of N elements in $\mathcal{A}_0$ is given. Then, for any $\alpha_n \in \mathbb{R}$, $n = 1\cdots N$, and any $u\in\mathcal{A}$,

$$u^N(x,t) = u(x,t) + \sum_{k=1}^{N} \alpha_k u_k(x,t), \quad (x,t)\in Q_T$$

still belongs, in the linear situation of Section 2, to $\mathcal{A}$. The minimization of $\|u^N(1,t)\|_{L^2(0,T)}$ is then reduced to a quadratic minimization on $\{\alpha_k\}_{k=1,N}$. The method we propose to construct the family $\{u_k\}_{k\in[1,N]}$ is as follows: we first compute N elements v_k, $k = 1,\cdots,N$ in $\mathcal{A}_0$ using the conjugate gradient algorithm with initial guesses $u_k^0(x,t) = x\sin(k\pi t/T)^2$: then, we orthogonalize

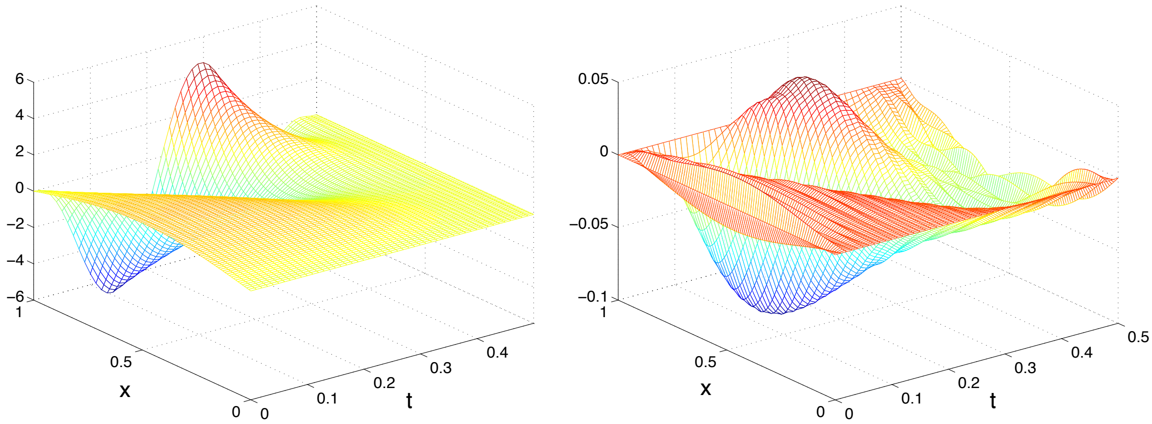

FIGURE 10. $u_0(x) = \sin(\pi x)$ - $F(s) = -5s\log(1+|s|)$, $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Control acting on $x = 0$ - $\varepsilon = 10^{-5}$- Solution in $u_h \in \mathcal{A}_h$ (**Left**) and gap $u_h - \overline{u}_h$ (**Right**) along Q_T. $E_h(u_h^{n=2\,788}) \approx 3.33 \times 10^{-6}$, $\|g_h^{n=2\,788}\|_{\mathcal{A}} \approx 9.89 \times 10^{-6}$ and $\|\overline{u}_h(\cdot,T)\|_{L^2(0,1)} \approx 1.92 \times 10^{-2}$.

these elements using the Gram-Schmidt procedure with the scalar product associated with $\mathcal{A}$:

$$u_k = v_k - \sum_{n=1}^{k-1} < v_k, u_n >_{\mathcal{A}} u_n.$$

Figure 11 shows the trace of $u^N \in \mathcal{A}$ along Σ_T obtained with $N = 10$ as well as the trace of $u \in \mathcal{A}$ corresponding to $u^0(x) = \sin(\pi x)(1-t/T)^2$ (see Figure 2). We obtain $\|u_h^N\|_{L^2(\Sigma_T)} \approx 0.981$ lower than $\|u_h\|_{L^2(\Sigma_T)} \approx 1.392$. Larger values of N, which require a finer mesh in time so as to capture the oscillating functions $\sin(m\pi t/T)$, do not allow a significant additional reduction of the $L^2(\Sigma_T)$-norm.

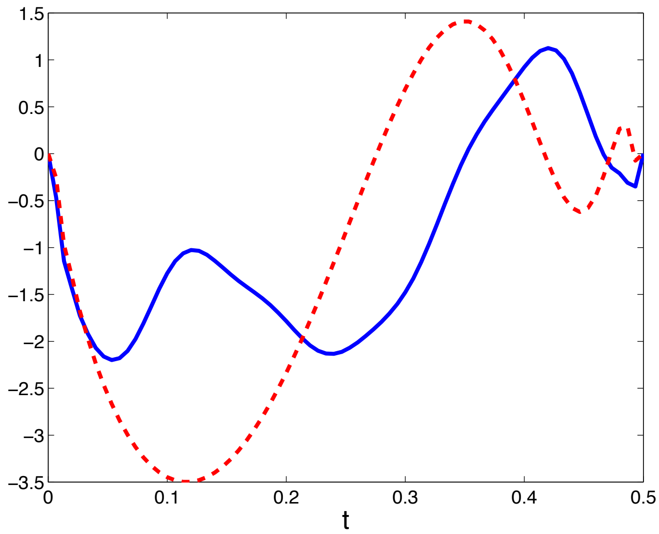

FIGURE 11. $u_0(x) = \sin(\pi x)$ - $T = 1/2$, $a_0 = 1/4$, $\Delta x = \Delta t = 1/100$ - Trace of u_h^N (**full line**) and of u_h (**dashed line**) along Σ_T.

This constructive approach which allows to jump from a local minimum of E to another one does not apply for the nonlinear situation of Section 5. On the other hand, the more flexible approach which consists to minimize at the same time the error functional E and the L^2-norm of the trace with respect to u, does not lead to satisfactory results, as it depends too much on the initial guess u^0. In that respect, a possible strategy could be to initialize the CG algorithm with an approximate control obtained from the dual approach (see [3]).

7. Concluding remarks

The variational approach we discussed here to construct numerical controls is very different in nature from the usual one [3, 8, 20] making use of dual variable to deal with the constraint $u(\cdot, T) = 0$. In the context of parabolic equations, this difference is significant because the variational approach avoids the approximation of singular functional spaces and therefore ill-posed problems. Here, the problem is elliptic and leads to standard and well-posed formulations. A quantitative comparison with the dual approach for the boundary situation remains however to be done. The method extends to any target - trajectory for the heat equation -, to higher dimensions, and to any system for which a unique continuation property is known.

It is also remarkable to note that this variational approach allows to solve inverse problems. Let us mention, in particular, the highly ill-posed *backward heat problem* which consists to determine the solution of the heat equation at time $t = 0$ from the solution u_T at any positive time T (we refer to [11]). It suffices to define the functional spaces $\mathcal{A}$ and $\mathcal{A}_0$ respectively as follows : $\mathcal{A} = \{u \in H^1(Q_T), u(0,t) = u(1,t) = 0, u(x,T) = u_T(x), (x,t) \in Q_T\}$ and $\mathcal{A}_0 = \{u \in H^1(Q_T), u(0,t) = u(1,t) = 0, u(x,T) = 0, (x,t) \in Q_T\}$. We plan to analyze this situation in a near future.

References

[1] G. Alessandrini and L. Escauriaza, *Null-controllability of one-dimensional parabolic equations,* ESAIM Control Optim. Calc. Var. 14 (2008), no. 2, 284–293.

[2] F. Boyer, F. Hubert and J. Le Rousseau, *Uniform null-controllability properties for space/time-discretized parabolic equations,* Preprint 2009.

[3] C. Carthel, R. Glowinski and J.-L. Lions, *On exact and approximate Boundary Controllability for the heat equation: A numerical approach,* J. Optimization, Theory and Applications 82(3), (1994) 429–484.

[4] J. M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, AMS, Vol. 136, 2007.

[5] S. Ervedoza and J. Valein, *On the observability of abstract time-discrete linear parabolic equations,* Rev. Mat. Comput. 23(1) (2010), 163–190.

[6] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations,* Ann. Inst. H. Poincaré Anal. Non Linéaire 17(5) (2000), 583–616.

[7] E. Fernández-Cara and A. Münch, *Numerical null controllability of the 1D heat equation: primal algorithms.* Preprint (2009).

[8] E. Fernández-Cara and A. Münch, *Numerical null controllability of the 1D heat equation: dual algorithms,* Preprint (2010).

[9] H.O. Fattorini and D.L. Russel, *Exact controllability theorems for linear parabolic equation in one space dimension*, Arch. Rational Mech. **43** (1971) 272-292.

[10] A.V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, number 34. Seoul National University, Korea, (1996) 1–163.

[11] D.N. Hào, *Methods for inverse heat conduction problems* Methods and Procedures in Mathematical Physics, 43. Peter Lang, Frankfurt am Main, 1998.

[12] S. Kindermann, *Convergence Rates of the Hilbert Uniqueness Method via Tikhonov regularization*, J. of Optimization Theory and Applications 103(3), (1999) 657-673.

[13] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems* Springer series in computational physics 1983.

[14] R. Glowinski, J.L. Lions and J. He, *Exact and approximate controllability for distributed parameter systems: a numerical approach* Encyclopedia of Mathematics and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[15] S. Labbé and E. Trélat, *Uniform controllability of semi-discrete approximations of parabolic control systems,* Systems and Control Letters 55 (2006) 597-609.

[16] B. Laroche, P. Martin and P. Rouchon, *Motion planning for the heat equation*, Int. Journal of Robust and Nonlinear Control **10**, (2000) 629-643.

[17] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Comm. Partial Differential Equations 20 (1995), no. 1–2, 335–356.

[18] J. L. Lions, *Optimal Control of Systems governed by Partial Differential Equations*, Springer, 1971.

[19] J. L. Lions, *Exact controllability, stabilizability and perturbations for distributed systems,* SIAM Rev., 30 (1988), 1-68.

[20] A. Münch and E. Zuazua, *Numerical approximation of null controls for the heat equation: ill-posedness and remedies,* Inverse Problems 26 (2010) no. 8 085018, 39pp.

[21] P. Pedregal, *A variational perspective on controllability,* Inverse Problems 26 (2010) no. 1, 015004, 17pp.

[22] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions*, SIAM Review, 20 (1978), 639-739.

[23] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, ICM2006, Madrid, Spain, Vol. III (2006) 1389–1417.