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Chapitre 1

Introduction

Ce chapitre est constitué tout d'abord (Section 1.1) d'une brève introduction à la théorie cinétique et aux modèles étudiés dans cette thèse. Ensuite, la Section 1.2 est consacrée à une présentation des principaux résultats développés dans les différentes parties de cette thèse.

1.1 Rappels sur quelques modèles de base en théorie cinétique

1.1.1 L'équation de Boltzmann

L'équation de base de la théorie cinétique est l'équation de Boltzmann, qui modélise un système composé d'un grand nombre de particules identiques soumis à des collisions binaires élastiques. Le système de particules est décrit à l'aide d'une fonction $f(t,x,v)$ appelée fonction de distribution, qui dépend du temps $t \geq 0$, de la position $x \in \mathbb{R}^3$ et de la vitesse $v \in \mathbb{R}^3$ des particules. Le problème des conditions aux bords ne sera pas abordé ici. Pour tout temps t, $f(t,x,v)\,dx\,dv$ représente le nombre probable de particules dans l'élément de volume $dx\,dv$ centré en (x,v). En l'absence de champ de force extérieur agissant sur les particules, f vérifie l'équation de Boltzmann (cf. [16, 17, 56])

$$\partial_t f + v\cdot\nabla_x f = Q_B(f,f), \tag{1.1.1}$$

avec

$$Q_B(f,f)(t,x,v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\big(f'f'_* - ff_*\big)\,d\omega\,dv_*, \tag{1.1.2}$$

où B est une fonction positive que l'on précisera par la suite et où les notations f, f_*, f' et f'_* désignent la même fonction f prise en différentes variables, i.e., $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$. Les vitesses v et v_* représentent les vitesses des particules avant collision tandis que v' et v'_* désignent les vitesses après collision. L'élasticité des collisions se traduit, au niveau microscopique, par la conservation de la quantité de mouvement et de l'énergie cinétique du système constitué des deux particules qui collisionnent. Plus précisément, on a

$$v + v_* = v' + v'_*, \tag{1.1.3}$$
$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2. \tag{1.1.4}$$

Ces équations forment un système de quatre équations à six inconnues. Les vitesses post-collisionnelles peuvent donc être décrites en fonction des vitesses pré-collisionnelles et d'un paramètre $\omega \in \mathbb{S}^2$ de la manière suivante :

$$v' = v - \big[(v-v_*)\cdot\omega\big]\,\omega, \qquad \text{et} \qquad v'_* = v_* + \big[(v-v_*)\cdot\omega\big]\,\omega. \tag{1.1.5}$$

La géométrie de la collision est résumée par la Figure 1.1.

L'opérateur (1.1.2) est un opérateur quadratique qui n'agit que sur la dépendance en vitesse de la fonction f. Il décrit les interactions entre les particules et peut être interprété de la manière suivante. On le sépare formellement en deux parties, un terme de gain et un terme de perte :

$$Q_B(f,f) = Q_B^+(f,f) - Q_B^-(f,f),$$

avec

$$Q_B^+(f,f)(t,x,v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega) f'f'_*\,d\omega\,dv_*,$$
$$Q_B^-(f,f)(t,x,v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega) ff_*\,d\omega\,dv_*.$$

Le terme de gain Q_B^+ correspond aux particules qui acquièrent la vitesse v lors d'une collision alors que le terme de perte Q_B^- correspond aux particules de vitesse v qui perdent cette vitesse au cours d'une collision.

La section efficace

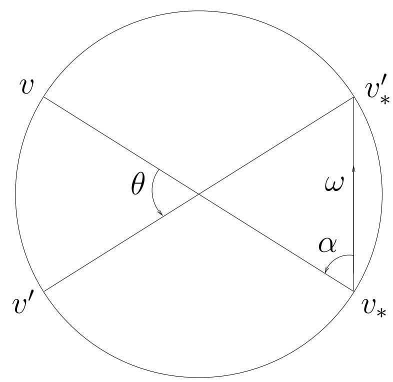

Fig. 1.1 – Représentation des vitesses dans l'espace des phases

La fonction $B(z,\omega)$ est une fonction mesurable positive, appelée section efficace de collision qui ne dépend que de $|z|$ et de $|z\cdot\omega|$. Etant données deux particules de vitesses v et v_* qui collisionnent, $B(v-v_*,\omega)$ peut être interprété comme une mesure de probabilité sur tous les choix possibles du paramètre ω. On s'intéresse principalement à deux interactions de natures différentes : le modèle des sphères dures et le cas des potentiels intermoléculaires.

Dans le modèle des sphères dures, les particules sont supposées rebondir les unes sur les autres comme des boules de billard. On peut alors déterminer de manière explicite la section efficace (cf. [16, 18]), qui s'écrit

$$B(z,\omega) = |z\cdot\omega|.$$

Dans le cas des potentiels intermoléculaires, deux particules se repoussent par des forces proportionnelles à $1/r^s$ avec $s \geq 2$, r désignant la distance entre les particules. La section efficace de collision s'écrit alors (cf. [16, 18])

$$B(z,\omega) = |z|^\gamma\, b(\theta) \quad \text{avec} \quad \theta = \pi - 2\,\mathrm{Arccos}\left(\frac{|z\cdot\omega|}{|z|}\right) \quad \text{et} \quad \gamma = \frac{s-5}{s-1}, \qquad (1.1.6)$$

où θ désigne l'angle de déviation relative de la particule lors de la collision (cf. Figure 1.1). Il est d'usage de classer les potentiels selon la valeur de l'exposant γ. On distingue alors les potentiels durs ($0<\gamma<1$), les potentiels mous ($-3<\gamma<0$), le potentiel maxwellien ($\gamma=0$) et le potentiel coulombien ($\gamma=-3$). La fonction b est régulière sauf en $\theta=0$ où elle présente une singularité,

$$\sin(\theta/2)\, b(\theta) \overset{\theta\to 0}{\sim} C\, \theta^{(\gamma-3)/2}. \qquad (1.1.7)$$

En particulier, dans le cas coulombien, on a

$$b(\theta) = \frac{C}{\sin^4(\theta/2)}.$$

La fonction b étant non intégrable en 0, il faut pouvoir donner un sens à l'intégrale de (1.1.2). Une hypothèse classique permet d'éviter cette difficulté en tronquant B et en supposant, par exemple, que B est intégrable par rapport à la variable angulaire. C'est l'hypothèse de troncature angulaire de Grad. Dans le cas des fonctions f régulières, l'expression $f'f'_* - ff_*$ est d'ordre 2 en θ et l'intégrale sur $\mathbb{S}^2$ est alors bien définie, sauf pour le potentiel coulombien. Sans hypothèse de régularité sur f, l'utilisation d'une formulation faible de (1.1.1) permet, de manière similaire, de donner un sens à l'intégrale sur $\mathbb{S}^2$ dans le cas non coulombien.

Les estimations *a priori*

Rappelons maintenant quelques propriétés formelles bien connues de l'équation de Boltzmann. Le changement de variables qui consiste à échanger les vitesses pré-collisionnelles et post-collisionnelles $(v, v_*) \to (v', v'_*)$ est involutif et de jacobien 1. Par conséquent, pour toute fonction test φ, on a

$$\int_{\mathbb{R}^3} Q_B(f,f)(v)\,\varphi(v)\,dv = \frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\big(f'f'_* - ff_*\big)\big(\varphi + \varphi_* - \varphi' - \varphi'_*\big) d\omega\, dv_*\, dv.$$

On déduit alors de la conservation de la quantité de mouvement (1.1.3) et de l'énergie cinétique (1.1.4) que

$$\int_{\mathbb{R}^3} Q_B(f,f)(v)\,\varphi(v)\,dv = 0 \qquad \text{pour} \quad \varphi(v) = 1, v, |v|^2.$$

Une des principales propriétés de l'équation de Boltzmann est son irréversibilité au niveau macroscopique. Au niveau microscopique, les collisions sont réversibles. Cependant, au niveau macroscopique, l'entropie de la fonction f, définie par

$$S(f) = \int_{\mathbb{R}^3} f(v)\ln(f(v))\,dv, \tag{1.1.8}$$

se dissipe. En effet, en multipliant l'équation de Boltzmann (1.1.1) par $\ln(f)$ et en intégrant par rapport aux vitesses, on obtient, formellement, l'équation locale de dissipation de l'entropie

$$\partial_t S(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v\, f(v)\ln(f(v))\,dv = D_B(f),$$

où

$$\begin{aligned} D_B(f) &= \int_{\mathbb{R}^3} Q_B(f,f)(v)\,\ln(f(v))\,dv \\ &= -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\big(f'f'_* - ff_*\big)\ln\left(\frac{f'f'_*}{ff_*}\right) d\omega\, dv_*\, dv. \end{aligned}$$

Comme $(x-y)\ln(x/y) \geq 0$ pour tout $x, y \geq 0$, on a $D_B(f) \leq 0$, avec égalité si et seulement si f est une maxwellienne

$$M(v) = \exp(a + b \cdot v - c|v|^2), \tag{1.1.9}$$

où $a \in \mathbb{R}$, $b \in \mathbb{R}^3$ et $c \in \mathbb{R}_+$. Plus précisément, si f est une fonction positive de $L^1(\mathbb{R}^3)$, on a (cf. [16]) l'équivalence suivante

$$Q_B(f,f) = 0 \iff D_B(f) = 0 \iff f(v) = \exp(a + b \cdot v - c|v|^2).$$

Alors, $-D_B(f)$ peut être interprété formellement comme une sorte de distance à l'ensemble des maxwelliennes. Les coefficients a, b et c de (1.1.9) peuvent être exprimés en fonction de grandeurs macroscopiques. Plus précisément, la maxwellienne (1.1.9) peut être écrite sous la forme

$$M(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-|v-u|^2/(2T)},$$

où la densité ρ, la vitesse macroscopique u et la température T sont données par

$$\rho = \int_{\mathbb{R}^3} M(v)\, dv, \qquad \rho u = \int_{\mathbb{R}^3} v\, M(v)\, dv \qquad \text{et} \qquad \rho|u|^2 + 3\rho T = \int_{\mathbb{R}^3} |v|^2\, M(v)\, dv.$$

L'irréversibilité de l'équation de Boltzmann se traduit de plus par la convergence (en temps grands) des solutions du problème de Cauchy associé à (1.1.1) vers une maxwellienne.

1.1.2 L'équation de Landau

Le modèle de Boltzmann est adapté aux cas des particules neutres ou de plasmas faiblement ionisés où les interactions charge-neutre sont les plus fréquentes. Dans le cas des plasmas complètement ionisés, en raison de la décroissance relativement lente de la force de Coulomb avec la distance, les interactions lointaines correspondant aux interactions entre particules chargées sont dominantes. Elles correspondent à de petits angles de déviation, c'est à dire que $(v - v_*) \cdot \omega$ est proche de 0 (collisions rasantes). Cela résulte en une forte singularité de la section efficace. Il est alors nécessaire d'introduire un nouveau modèle, celui de Landau (cf. [18, 56]), qui permet de tenir compte de l'effet de ces collisions dominantes et s'écrit

$$\partial_t f + v \cdot \nabla_x f = Q_L(f,f), \tag{1.1.10}$$

où l'opérateur de collision est donné par

$$Q_L(f,f)(t,x,v) = \nabla_v \cdot \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \big(f(v_*) \nabla f(v) - f(v) \nabla f(v_*)\big)\, dv_*, \tag{1.1.11}$$

avec

$$\Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}, \qquad 1 \leq i,j \leq 3. \tag{1.1.12}$$

Le cas $\gamma = -3$ correspond à l'interaction coulombienne, c'est le cas le plus utilisé en physique, notamment pour la modélisation des systèmes de particules chargées (plasmas). Cette équation se généralise au cas où $-3 < \gamma \leq 1$. Comme pour l'équation de Boltzmann, on distingue alors le potentiel coulombien $\gamma = -3$, les potentiels mous $-3 < \gamma < 0$, le potentiel maxwellien $\gamma = 0$ et les potentiels durs $0 < \gamma \leq 1$.

Les estimations *a priori*

L'équation de Landau vérifie des propriétés similaires à celles satisfaites par l'équation de Boltzmann. Pour toute fonction test φ, on a

$$\int_{\mathbb{R}^3} Q_L(f,f)(v)\,\varphi(v)\,dv = -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^{\gamma+2}\,\Pi(v-v_*)\big(f_*\nabla f - f\nabla f_*\big)\,\big(\nabla\varphi-\nabla\varphi_*\big)\,dv_*\,dv.$$

Comme $\Pi(z)$ est la projection orthogonale sur $(\mathbb{R}z)^\perp$, on déduit que

$$\int_{\mathbb{R}^3} Q_L(f,f)(v)\,\varphi(v)\,dv = 0 \qquad \text{pour} \qquad \varphi(v) = 1, v, |v|^2.$$

Comme pour l'équation de Boltzmann, on définit l'entropie de f par (1.1.8). On déduit alors l'équation locale de dissipation de l'entropie

$$\partial_t S(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v\, f(v) \ln(f(v))\,dv = D_L(f).$$

où

$$\begin{aligned} D_L(f) &= \int_{\mathbb{R}^3} Q_L(f,f)(v)\,\ln(f(v))\,dv,\\ &= -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^{\gamma+2}\,\Pi(v-v_*)\,f f_*\left(\frac{\nabla f}{f}-\frac{\nabla f_*}{f_*}\right)\left(\frac{\nabla f}{f}-\frac{\nabla f_*}{f_*}\right)dv_*\,dv.\end{aligned}$$

Comme la matrice Π est semi-définie positive, on a $D_L(f) \le 0$.

L'asymptotique des collisions rasantes

L'équation de Landau peut être vue comme une approximation de l'équation de Boltzmann lorsque les collisions rasantes sont prédominantes. D'après (1.1.6)-(1.1.7), dans le cas des potentiels intermoléculaires, lorsque la section efficace de collision B est exprimée en fonction de θ, elle présente une singularité en $\theta = 0$, ce qui correspond aux faibles déviations, c'est à dire aux collisions rasantes. Dans le cas des particules chargées, ces collisions rasantes sont dominantes. Ainsi, dans le cas coulombien, l'intégrale en θ diverge de manière logarithmique, ce qui a amené Degond et Lucquin-Desreux [23] à considérer l'opérateur

$$Q_B^\varepsilon(f,f)(v) = \int_{\mathbb{R}^3}\int_\varepsilon^\pi\int_0^{2\pi} \frac{1}{|v-v_*|^3}\,\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\,(f'f'_* - f f_*)\,d\varphi\,d\theta\,dv_*, \tag{1.1.13}$$

qui correspond à une troncature du cas coulombien. En effectuant un développement asymptotique de $f'f'_* - ff_*$ lorsque $\theta \to 0$, ils ont montré que, pour une fonction f suffisamment régulière, on a

$$Q_B^\varepsilon(f,f)(v) = \pi\,|\ln(\varepsilon)|\,\nabla_v \cdot \int_{\mathbb{R}^3} \frac{1}{|v-v_*|}\Pi(v-v_*)\,\big(f_*\nabla f - f\nabla f_*\big)\,dv_* + O(1),$$

quand ε tend vers 0. On retrouve donc l'opérateur de Landau en première approximation.

Dans le cas non coulombien, Desvillettes [24] a considéré une asymptotique de l'opérateur (1.1.2) quand la section efficace de collision B se concentre autour de $\theta = 0$. Il a introduit la famille de sections efficaces

$$\tilde{B}^\varepsilon(|v-v_*|,\theta) = \frac{1}{\varepsilon^3}\,\tilde{B}\left(|v-v_*|,\frac{\theta}{\varepsilon}\right),$$

et la famille d'opérateurs de collision associée

$$Q_B^\varepsilon(f,f)(v) = \int_{\mathbb{R}^3}\int_0^\pi\int_0^{2\pi} \tilde{B}^\varepsilon(|v-v_*|,\theta)(f'f'_* - ff_*)\,d\varphi\,d\theta\,dv_*, \tag{1.1.14}$$

où $\tilde{B}$ désigne la fonction définie par $\tilde{B}(|v-v_*|,\theta) = \sin\theta B(v-v_*,\omega)$ et prolongée par 0 sur $\mathbb{R}_+ \times \mathbb{R}$. En effectuant un développement asymptotique de $f'f'_* - ff_*$ lorsque $\theta \to 0$, Desvillettes a alors obtenu, pour une fonction f et une section efficace $\tilde{B}$ suffisamment régulières que

$$Q_B^\varepsilon(f,f)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \lambda(|v-v_*|)|v-v_*|^2\,\Pi(v-v_*)(f_*\nabla f - f\nabla f_*)\,dv_* + O(\varepsilon)$$

quand ε tend vers 0, avec

$$\lambda(|z|) = \frac{\pi}{8}\int_0^\pi \theta^2\tilde{B}(|z|,\theta)\,d\theta.$$

Dans le cas des forces en $1/r^s$ décrit dans (1.1.6), on retrouve l'opérateur de Landau.

Plus récemment, cette asymptotique développée séparément par Degond et Lucquin-Desreux d'une part et Desvillettes d'autre part a été rendue plus précise. En effet, il a été démontré que les solutions des équations de Boltzmann correspondant aux opérateurs (1.1.13) et (1.1.14) convergent vers une solution de l'équation de Landau, aussi bien dans le cas spatialement homogène [41, 70] que dans dans le cas inhomogène en espace [4].

La théorie mathématique des équations de Boltzmann et Landau est très avancée. Elle comporte notamment des résultats d'existence, d'unicité, de propagation de moments, et de retour vers l'équilibre, ... Nous ne détaillerons pas ces résultats ici. Le lecteur intéressé pourra consulter, par exemple, [16, 17, 71] et les références incluses.

1.1.3 Variantes de l'équation de Boltzmann

L'équation de Boltzmann quantique

Dans certains cas (gaz constitués de particules légères, basse température), les effets quantiques ne sont pas négligeables et doivent être inclus dans la description (cf. [18]). Il ne s'agit pas d'utiliser une description quantique à l'aide d'une fonction d'onde ou d'une transformée de Wigner. Les effets quantiques sont pris en compte au niveau de l'opérateur de collision. Pour être plus précis, supposons tout d'abord que le gaz est constitué de particules de Fermi-Dirac. Ces particules doivent satisfaire au principe d'exclusion de

Pauli. Par conséquent, une collision entre deux particules ne peut avoir lieu que si elle fait passer les particules dans des états qui ne sont pas déjà occupés. Si, au contraire, on considère des particules de Bose-Einstein, une collision entre deux particules qui fait passer ces deux particules dans des états qui sont déjà occupés a plus de chance d'avoir lieu. L'équation de Boltzmann quantique s'écrit

$$\partial_t f + v \cdot \nabla_x f = Q_{BQ}(f), \tag{1.1.15}$$

où

$$Q_{BQ}(f)(t,x,v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\Big\{f'f'_*(1-\delta f)(1-\delta f_*)-ff_*(1-\delta f')(1-\delta f'_*)\Big\}\,d\omega\,dv_*. \tag{1.1.16}$$

Comme pour l'équation de Boltzmann, les vitesses v' et v'_* sont données par (1.1.5) qui constitue une paramétrisation des solutions du système constitué de (1.1.3) et (1.1.4). On a conservé ici les notations f, f_*, f' et f'_* introduites dans la Section 1.1.1. La section efficace B vérifie les mêmes propriétés que celle de l'équation de Boltzmann classique (cf. Section 1.1.1).

Le paramètre δ vaut 0, 1 ou -1 (après renormalisation). Le cas $\delta = 0$ correspond au cas classique de l'équation de Boltzmann. Les cas $\delta = 1$ et $\delta = -1$ correspondent respectivement aux cas des particules de Fermi-Dirac et de Bose-Einstein.

Rappelons maintenant quelques propriétés formelles de (1.1.15), certaines étant similaires à celles de l'équation de Boltzmann classique. En multipliant (1.1.16) par une fonction test φ et en effectuant les mêmes opérations que pour l'opérateur de Boltzmann classique, on obtient

$$\int_{\mathbb{R}^3} Q_{BQ}(f)(v)\,\varphi(v)\,dv = \frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\left(\varphi+\varphi_*-\varphi'-\varphi'_*\right) \times \left(f'f'_*(1-\delta f)(1-\delta f_*)-ff_*(1-\delta f')(1-\delta f'_*)\right)d\omega\,dv_*\,dv.$$

Puisque les vitesses v, v_*, v' et v'_* satisfont (1.1.3) et (1.1.4), il vient que

$$\int_{\mathbb{R}^3} Q_{BQ}(f)(v)\,\varphi(v)\,dv = 0 \qquad \text{pour} \qquad \varphi(v) = 1, v, |v|^2.$$

Dans le cas des particules de Fermi-Dirac ($\delta = 1$), le principe d'exclusion de Pauli implique qu'une solution de (1.1.15)-(1.1.16) vérifie $0 \leq f \leq 1$ dès que la condition initiale vérifie cette borne. Dans ce cas, l'entropie est définie par

$$S_{FD}(f) = -\int_{\mathbb{R}^3}\big(f(v)\ln(f(v)) + (1-f(v))\ln(1-f(v))\big)\,dv. \tag{1.1.17}$$

En raison de la borne sur f, l'entropie est positive (le choix du signe moins a été fait de manière à manipuler une quantité positive). De plus, il suffit que la fonction f soit d'énergie finie pour que l'entropie soit finie. En effet, puisque la fonction $s : r \mapsto r\,|\ln r| + (1-r)\,|\ln(1-r)|$ est concave, on a l'inégalité

$$s(f(v)) \leq f(v)\,|v|^2 + e^{-|v|^2},$$

pour tout $v \in \mathbb{R}^3$. La multiplication de (1.1.16) par $\ln(f) - \ln(1-f)$ et l'intégration par rapport à la vitesse conduisent à l'équation de dissipation de l'entropie

$$\partial_t S_{FD}(f) - \nabla_x \cdot \int_{\mathbb{R}^3} v\, f(v)\big(\ln(f(v)) - \ln(1-f(v))\big)\, dv = D_{BFD}(f),$$

où

$$\begin{aligned} D_{BFD}(f) &= \int_{\mathbb{R}^3} Q_{BFD}(f)(v)\big(\ln(f(v)) - \ln(1-f(v))\big)\, dv \\ &= \frac{1}{4} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\, \Gamma\big(f' f'_*(1-f)(1-f_*), f f_*(1-f')(1-f'_*)\big)\, d\omega\, dv_*\, dv, \end{aligned}$$

avec $\Gamma(x,y) = (x-y)\ln(x/y)$. Comme pour l'équation de Boltzmann, on déduit que $D_{BFD}(f) \geq 0$. De plus, on a égalité si et seulement si f est une distribution de Fermi-Dirac, c'est à dire de la forme

$$f(v) = \frac{a e^{-b|v-v_0|^2}}{1 + a e^{-b|v-v_0|^2}},$$

avec $a, b > 0$ et $v_0 \in \mathbb{R}^3$ ou si f est une fonction caractéristique d'une boule de $\mathbb{R}^3$ (cf. [59]).

L'équation de Boltzmann pour les particules de Fermi-Dirac a fait, pour l'instant, l'objet de peu de travaux. Dans le cas spatialement inhomogène, des théorèmes d'existence ont été obtenus par Dolbeault [28] sous l'hypothèse que $B \in L^1(\mathbb{R}^3 \times \mathbb{S}^2)$ et Lions [57] dans le cas où $B(z,\theta)$ est intégrable en θ et localement intégrable en z. Finalement, Alexandre [2] a montré, dans le cas spatialement inhomogène, l'existence de solutions pour les potentiels durs et les potentiels mous en tronquant B pour des vitesses relatives petites. Le cas spatialement homogène a été considéré par Lu [59] pour des sections efficaces de collision de la forme $B(z,\omega) = b(\theta)|z|^\gamma$ où $-3 < \gamma \leq 1$ et

$$\int_0^{\pi/2} \sin(\theta)\, b(\theta)\, d\theta < \infty.$$

La caractérisation des états d'équilibre et le retour vers l'équilibre en temps grand ont également été traités récemment par Lu et Wennberg [59, 61].

Dans le cas des particules de Bose-Einstein ($\delta = -1$), l'entropie est définie par

$$S_{BE}(f) = \int_{\mathbb{R}^3} \big(f(v)\ln(f(v)) - (1+f(v))\ln(1+f(v))\big)\, dv.$$

La multiplication de (1.1.16) par $\ln(f) - \ln(1+f)$ et l'intégration par rapport à la vitesse conduit à l'équation de dissipation de l'entropie

$$\partial_t S_{BE}(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v\, f(v)\big(\ln(f(v)) - \ln(1+f(v))\big)\, dv = D_{BBE}(f),$$

où

$$\begin{aligned} D_{BBE}(f) &= \int_{\mathbb{R}^3} Q_{BBE}(f)(v)\big(\ln(f(v)) - \ln(1+f(v))\big)dv \\ &= -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\omega)\,\Gamma\big(f'f'_*(1+f)(1+f_*), ff_*(1+f')(1+f'_*)\big)\,d\omega\,dv_*\,dv \\ &\leq 0. \end{aligned}$$

Formellement, si f est une fonction régulière et si $D_{BBE}(f) = 0$, alors f est une distribution de Bose-Einstein, c'est à dire de la forme

$$f(v) = \frac{ae^{-b|v-v_0|^2}}{1 - ae^{-b|v-v_0|^2}},$$

avec $0 \leq a \leq 1$, $b > 0$ et $v_0 \in \mathbb{R}^3$. Contrairement au cas classique et au cas des particules de Fermi-Dirac, la distribution de Bose-Einstein présente une singularité dans le cas où $a = 1$.

Comme pour l'équation de Boltzmann-Fermi-Dirac, l'équation de Boltzmann-Bose-Einstein présente une forte non-linéarité. En revanche, contrairement à l'équation de Boltzmann-Fermi-Dirac, les solutions de l'équation de Boltzmann-Bose-Einstein ne satisfont pas de borne L^∞. Elles ne satisfont pas non plus de borne $L \log L$ puisque l'entropie est sous-linéaire. Tout ceci complique l'analyse. Pour l'instant, seul le cas des solutions isotropes a été étudié [58, 35, 60].

L'équation de Boltzmann relativiste

Dans le cas où la vitesse des particules n'est pas négligeable par rapport à la vitesse de la lumière c, la mécanique classique doit être remplacée par la mécanique relativiste. L'équation de Boltzmann se généralise au cas de la relativité restreinte [39, 44, 51]. Une particule relativiste de masse m et d'impulsion p a pour énergie

$$\varepsilon(p) = c\sqrt{m^2c^2 + |p|^2} = \gamma\, m\, c^2, \qquad \text{avec} \qquad \gamma = \gamma(p) = \sqrt{1 + \frac{|p|^2}{m^2c^2}},$$

et pour vitesse

$$v(p) = \nabla_p\, \varepsilon(p) = \frac{p}{m\,\gamma(p)}. \tag{1.1.18}$$

Considérons une collision élastique de deux particules d'impulsions p et p_* et notons p' et p'_* les impulsions de ces deux particules après collision. On a, comme dans le cas classique, conservation de l'impulsion

$$p + p_* = p' + p'_*, \tag{1.1.19}$$

et de l'énergie

$$\varepsilon(p) + \varepsilon(p_*) = \varepsilon(p') + \varepsilon(p'_*). \tag{1.1.20}$$

Ces équations forment un système de quatre équations à six inconnues. Les impulsions post-collisionnelles peuvent donc être décrites en fonction des impulsions pré-collisionnels et d'un paramètre $\omega \in \mathbb{S}^2$ de la manière suivante (cf. [35]) :

$$\begin{aligned} p' &= \frac{p+p_*}{2} + \frac{1}{2c}\sqrt{(\varepsilon(p)+\varepsilon(p_*))^2 - c^2|p+p_*|^2 - 4m^2c^4}\,\left(\omega + \frac{\gamma_V - 1}{|V|^2}\,VV^T\omega\right), \\ p'_* &= \frac{p+p_*}{2} - \frac{1}{2c}\sqrt{(\varepsilon(p)+\varepsilon(p_*))^2 - c^2|p+p_*|^2 - 4m^2c^4}\,\left(\omega + \frac{\gamma_V - 1}{|V|^2}\,VV^T\omega\right), \end{aligned}$$

où

$$V = \frac{(p+p_*)c}{\varepsilon(p)+\varepsilon(p_*)}, \qquad \text{et} \qquad \gamma_V = \frac{1}{\sqrt{1-|V|^2}}.$$

Si l'on considère un gaz composé de particules identiques, la fonction de distribution $(t,x,p) \mapsto f(t,x,p)$ associée à ce gaz vérifie l'équation de Boltzmann relativiste (cf. [39, 44, 51])

$$\partial_t f + v(p)\cdot\nabla_x f = Q_R(f,f), \tag{1.1.21}$$

avec

$$Q_R(f,f)(t,x,p) = \iint_{\mathbb{S}^2\times\mathbb{R}^3} \sigma\, v_M\,(f'f'_* - ff_*)\,dp_*\,d\omega,$$

où $f = f(t,x,p)$, $f_* = f(t,x,p_*)$, $f' = f(t,x,p')$, $f'_* = f(t,x,p'_*)$, σ désigne la section efficace, v_M la vitesse de Møller

$$v_M(p,p_*) = |v_{rel}|\,\frac{\varepsilon\varepsilon_* - c^2\,p\cdot p_*}{\varepsilon\varepsilon_*} = \left(|v-v_*|^2 - \frac{|v\times v_*|^2}{c^2}\right)^{1/2},$$

où $\varepsilon = \varepsilon(p)$, $\varepsilon_* = \varepsilon(p_*)$, $v = v(p)$ et $v_* = v(p_*)$ et où la vitesse relative v_{rel} de deux particules correspond, dans le cas relativiste, à la vitesse d'une particule dans le référentiel de repos de l'autre particule. Sa norme s'écrit

$$|v_{rel}| = \frac{\sqrt{|v-v_*|^2 - \frac{|v\times v_*|^2}{c^2}}}{1 - \frac{v\cdot v_*}{c^2}},$$

et est, en général, différente de $|v-v_*|$. Dans la limite classique $c \to +\infty$, la vitesse de Møller tend vers la vitesse relative classique $|v-v_*|$.

La forme de l'équation (1.1.21) est similaire à celle de l'équation de Boltzmann classique. Le caractère relativiste intervient dans la relation (1.1.18) qui lie l'impulsion à la vitesse et dans la définition de la vitesse de Møller. Elle intervient également implicitement dans la définition de la section efficace σ, qui est une fonction de

$$s = \frac{(\varepsilon+\varepsilon_*)^2}{c^2} - |p+p_*|^2,$$

et de l'angle de déviation θ (dans le référentiel du centre de masse) défini par

$$\cos\theta = \frac{(\varepsilon-\varepsilon_*)\left(\varepsilon'-\varepsilon'_*\right) - c^2\left(p-p_*\right)\cdot\left(p'-p'_*\right)}{\left(\varepsilon-\varepsilon_*\right)^2 - c^2\,|p-p_*|^2}.$$

L'équation de Boltzmann relativiste vérifie des estimations *a priori* similaires à celles satisfaites par l'équation de Boltzmann classique. Une des différences principales est que le jacobien de l'application $(p, p_*) \mapsto (p', p'_*)$ n'est pas égal à 1 dans le cas relativiste. Cependant, comme

$$v_M(p, p_*) = v_M(p', p'_*) \, \frac{\partial(p', p'_*)}{\partial(p, p_*)}$$

on obtient la formulation faible suivante

$$\int_{\mathbb{R}^3} Q_R(f,f)(p)\,\varphi(p)\,dp = \frac{1}{4} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \sigma\, v_M\, (f'f'_* - f f_*)(\varphi + \varphi_* - \varphi' - \varphi'_*)\,dp\,dp_*\,d\omega.$$

On déduit à l'aide de (1.1.19) et (1.1.20) que la masse, l'impulsion et l'énergie sont des quantités localement conservées.

En multipliant (1.1.21) par $\ln(f)$, on obtient la dissipation locale

$$\partial_t S(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v(p)\, f(p) \ln(f(p))\,dv = D_R(f),$$

de l'entropie

$$S(f) = \int_{\mathbb{R}^3} f(p) \ln(f(p))\,dp,$$

avec

$$\begin{aligned} D_R(f) &= \int Q_R(f,f)(p)\, \ln(f(p))\,dp \\ &= -\frac{1}{4} \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} \sigma v_M \big(f'f'_* - f f_*\big) \ln\left(\frac{f'f'_*}{f f_*}\right) d\omega\,dv_*\,dv. \end{aligned}$$

Comme pour l'équation de Boltzmann classique, on a $D_R(f) \le 0$. Formellement, $D_R(f) = 0$ si et seulement si f est une maxwellienne relativiste

$$f(p) = \exp(\alpha + \beta \cdot p - \gamma\, \varepsilon(p)),$$

où $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^3$ et $\gamma \in \mathbb{R}_+$.

L'équation de Boltzmann relativiste a pour l'instant été peu étudiée. On peut, par exemple, consulter [32, 40, 5, 45].

L'équation de Kac

Le modèle de Kac est un modèle simplifié de l'équation de Boltzmann. C'est un modèle non linéaire posé en dimension un et introduit par Kac [46]. Il considère un ensemble de N particules ponctuelles de vitesses unidimensionnelles $v_i \in \mathbb{R}$, $i = 1, \dots, N$ vérifiant

$$v_1^2 + \cdots + v_N^2 = N.$$

A des intervalles de temps exponentiellement distribués, deux particules sont sélectionnées. Elles collisionnent et leurs vitesses v_i et v_j prennent respectivement les valeurs v'_i et v'_j, où

v_i' et v_j' sont solutions de l'équation $v_i^2+v_j^2 = {v_i'}^2+{v_j'}^2$. Il y a ainsi conservation de l'énergie cinétique totale de l'ensemble des N particules. Les vitesses v_i' et v_j' sont obtenues à partir des vitesses v_i et v_j par une rotation de $\mathbb{R}^2$. On a

$$v_i' = v_i \cos\theta - v_j \sin\theta, \tag{1.1.22}$$

$$v_j' = v_i \sin\theta + v_j \cos\theta, \tag{1.1.23}$$

où θ est choisi à partir d'une distribution uniforme sur $(-\pi, \pi)$.

Notons Ψ_N la densité correspondant à l'ensemble des N particules ponctuelles évoluant suivant le modèle ci-dessus. Alors, Ψ_N vérifie

$$\partial_t \Psi_N(t, V) = K(\Psi_N)(t, V),$$

où $V = (v_1, \ldots, v_N) \in \mathbb{S}^{N-1}(\sqrt{N})$ et K est l'opérateur linéaire donné par

$$K(\Psi_N) = N(\tilde{K} - I)(\Psi_N), \tag{1.1.24}$$

l'opérateur I étant l'opérateur identité et $\tilde{K}$ étant défini par

$$\tilde{K}(\Psi_N)(t, V) = \binom{N}{2}^{-1} \sum_{1\le i<j\le N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi_N(t, R_{i,j}(\theta)V)\, d\theta,$$

avec $R_{i,j}(\theta)V = (v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N)$.

On définit alors

$$f_N^1(t, v_1) = \int_{\mathbb{S}^{N-2}(\sqrt{N-v_1^2})} \Psi_N(t, v_1, v_2, \ldots, v_N)\, d\sigma(v_2, \ldots, v_N), \tag{1.1.25}$$

où $d\sigma(v_2, \ldots, v_N)$ est l'élément de surface sur $\mathbb{S}^{N-2}(\sqrt{N-v_1^2})$. Kac a prouvé que, sous certaines conditions sur la donnée initiale, la fonction f_N^1 converge, lorsque N tend vers l'infini, vers une solution f de l'équation de Kac

$$\partial_t f(t, v) = Q_K(f, f)(t, v), \qquad t \in \mathbb{R}_+, v \in \mathbb{R}, \tag{1.1.26}$$

où l'opérateur de collision Q_K est donné par

$$Q_K(f, f)(t, v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f(t, v')f(t, v_*') - f(t, v)f(t, v_*)) \frac{1}{2\pi}\, d\theta\, dv_*. \tag{1.1.27}$$

En multipliant (1.1.27) par une fonction test φ, on obtient

$$\int_{\mathbb{R}} Q_K(f, f)(v)\, \varphi(v) dv = \frac{1}{8\pi} \iint_{\mathbb{R}\times\mathbb{R}} \int_{-\pi}^{\pi} (f' f_*' - f f_*)\, (\varphi + \varphi_* - \varphi' - \varphi_*')\, d\theta\, dv_*\, dv. \tag{1.1.28}$$

Il y a conservation de l'énergie au cours de chaque collision donc on déduit que

$$\int_{\mathbb{R}} Q_K(f, f)(v)\, \varphi(v)\, dv = 0 \qquad \text{pour} \quad \varphi(v) = 1, |v|^2.$$

On a donc conservation globale de la masse et de l'énergie

$$\frac{d}{dt}\int_{\mathbb{R}} f(t,v)\,dv = 0 \qquad \text{et} \qquad \frac{d}{dt}\int_{\mathbb{R}} f(t,v)\,|v|^2\,dv = 0.$$

En revanche, il n'y a, en général, pas conservation de la quantité de mouvement (sauf si elle est nulle initialement).

Le choix $\varphi = \ln f$ dans (1.1.28) conduit à

$$\begin{aligned} D_K(f) &= \int_{\mathbb{R}} Q_K(f,f)(v)\ \ln(f(v))dv \\ &= -\frac{1}{8\pi}\iint_{\mathbb{R}\times\mathbb{R}}\int_{-\pi}^{\pi}(f'f'_* - ff_*)\ \ln\left(\frac{f'f'_*}{ff_*}\right)\,d\theta\,dv_*\,dv\ \le\ 0. \end{aligned}$$

En multipliant (1.1.26) par $\ln f$ et en intégrant par rapport à la variable vitesse, on obtient la dissipation d'entropie

$$\frac{d}{dt}S(f) = D_K(f) \le 0, \qquad \text{avec} \qquad S(f) = \int_{\mathbb{R}} f(v)\ln(f(v))\,dv.$$

Les états d'équilibre de l'équation de Kac sont les maxwelliennes centrées $M(v) = ae^{-c|v|^2}$, où $a, c \in \mathbb{R}_+$.

L'existence de solutions de l'équation de Kac (1.1.26) a été démontrée dans [25] où Desvillettes a généralisé l'opérateur de collision de Kac au cas de sections efficaces non intégrables. Il a introduit l'opérateur

$$\tilde{Q}_K(f,f)(t,v) = \int_{\mathbb{R}}\int_{-\pi}^{\pi}(f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\,b(\theta)\,d\theta\,dv_*,$$

où

$$b(\theta) = |\theta|^{-1-\alpha}, \qquad \theta \in (-\pi,\pi), \qquad \alpha \in (0,2).$$

Des résultats d'existence, d'unicité, de régularité, de positivité et de retour vers l'équilibre ont été démontré pour cette équation de Kac sans "cut-off" (cf. [25, 38, 36]).

1.2 Analyse mathématique de quelques modèles issus de la théorie cinétique

1.2.1 L'équation de Landau-Fermi-Dirac

Tout comme l'équation de Boltzmann classique, l'équation de Boltzmann quantique (cf. Section 1.1.3) est adaptée aux cas des particules neutres ou de plasmas faiblement ionisés. En revanche, dans le cas des plasmas complètement ionisés, il faut tenir compte de la prépondérance des collisions rasantes. L'asymptotique des collisions rasantes décrite dans la Section 1.1.2 conduit alors à l'équation de Landau quantique

$$\partial_t f + v\cdot\nabla_x f = Q_{LQ}(f),$$

avec

$$Q_{LQ}(f)(t,x,v) = \nabla_v \cdot \int_{\mathbb{R}^3} |v-v_*|^{\gamma+2}\, \Pi(v-v_*) \Big\{ f_*(1-\delta f_*)\nabla f - f(1-\delta f)\nabla f_* \Big\}\, dv_*,$$

où la matrice Π est définie par (1.1.12). On rappelle que le cas $\delta = 0$ correspond au cas classique de l'équation de Landau et que les cas $\delta = 1$ et $\delta = -1$ correspondent respectivement aux cas des particules de Fermi-Dirac et de Bose-Einstein. Dans le cas $\delta = 1$, cette équation a également été introduite pour modéliser des particules auto-gravitantes [19, 47].

On ne considère ici que le cas des particules de Fermi-Dirac, c'est à dire le cas $\delta = 1$. On parle alors d'équation de Landau-Fermi-Dirac (LFD). Dans le Chapitre 2, on s'intéresse au problème de Cauchy pour l'équation de LFD dans le cas spatialement homogène. Elle s'écrit alors

$$\partial_t f = Q_{LFD}(f), \tag{1.2.1}$$

où

$$Q_{LFD}(f)(t,v) = \nabla_v \cdot \int_{\mathbb{R}^3} |v-v_*|^{\gamma+2}\, \Pi(v-v_*) \Big\{ f_*(1-f_*)\nabla f - f(1-f)\nabla f_* \Big\}\, dv_*. \tag{1.2.2}$$

L'équation (1.2.1) est complétée par la condition initiale

$$f(0) = f_{in}, \qquad \text{où} \qquad f_{in} \in L^1_2(\mathbb{R}^3), \qquad \text{et} \qquad 0 \le f_{in} \le 1 \ \text{ p.p.}, \tag{1.2.3}$$

où $L^1_2(\mathbb{R}^3) := L^1(\mathbb{R}^3; (1+|v|^2)\, dv)$. Comme pour l'équation de Boltzmann-Fermi-Dirac, le principe d'exclusion de Pauli implique que les solutions f de l'équation de LFD vérifient les bornes $0 \le f \le 1$ dès que cette condition est satisfaite par la donnée initiale.

On retrouve le même type d'estimations *a priori* que pour les équations de Boltzmann et Landau classiques. En effet, en multipliant (1.2.2) par une fonction test φ, on obtient

$$\begin{aligned}&\int_{\mathbb{R}^3} Q_{LFD}(f)(v)\, \varphi(v)\, dv \\ &= -\frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^{\gamma+2}\, \Pi(v-v_*) \big(f_*(1-f_*)\nabla f - f(1-f)\nabla f_* \big) \big(\nabla\varphi - \nabla\varphi_* \big)\, dv_*\, dv.\end{aligned}$$

Comme la matrice Π vérifie $\mathrm{Ker}(\Pi(z)) = \mathbb{R}z$, on déduit que

$$\int_{\mathbb{R}^3} Q_{LFD}(f)(v)\, \varphi(v)\, dv = 0 \qquad \text{pour} \qquad \varphi(v) = 1, v, |v|^2,$$

ce qui permet d'obtenir, formellement, la conservation de la masse, de la quantité de mouvement et de l'énergie.

Pour l'équation de LFD, l'entropie est donnée par (1.1.17) et le terme de dissipation d'entropie s'écrit alors

$$\begin{aligned} D_{LFD}(f) &= \int_{\mathbb{R}^3} Q_{LFD}(f) \Big[\ln(1-f) - \ln f \Big]\, dv \\ &= \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Pi(v-v_*) |v-v_*|^{\gamma+2} \\ &\qquad \Big(f_*(1-f_*)\nabla f - f(1-f)\nabla f_* \Big) \left(\frac{\nabla f}{f(1-f)} - \frac{\nabla f_*}{f_*(1-f_*)} \right) dv_*\, dv. \end{aligned} \tag{1.2.4}$$

La matrice Π est semi-définie positive, donc $D_{LFD}(f) \geq 0$. On obtient l'équation de dissipation de l'entropie

$$\frac{d}{dt} S_{FD}(f) = D_{LFD}(f) \geq 0.$$

Les Chapitres 2 et 3 correspondent respectivement aux articles [7] et [10]. Avant de présenter les résultats de ces chapitres, on introduit quelques notations. On définit, pour $(i,j) \in [\![1,3]\!]^2$,

$$a_{i,j}(z) = |z|^{\gamma+2} \left(\delta_{i,j} - \frac{z_i z_j}{|z|^2} \right), \qquad b_i(z) = \sum_k \partial_k a_{i,k}(z) = -2\,|z|^{\gamma}\, z_i,$$

$$\text{et} \qquad c(z) = \sum_{k,l} \partial^2_{kl} a_{k,l}(z) = -2\,(\gamma+3)\,|z|^{\gamma}.$$

Avec les notations

$$\overline{b}_i = b_i * f, \qquad \overline{c} = c * f, \qquad \overline{A}_{i,j} = a_{i,j} * \big(f(1-f)\big), \qquad \text{et} \qquad \overline{B}_i = b_i * \big(f(1-f)\big),$$

l'équation de Landau-Fermi-Dirac (1.2.1) s'écrit

$$\partial_t f = \sum_{i,j} \overline{A}_{i,j}\, \partial^2_{i,j} f + \big(\overline{B} - \overline{b}\,(1-2f)\big) \cdot \nabla f - \overline{c}\, f(1-f).$$

L'équation de LFD a pour l'instant été peu étudiée. Danielewicz [22] a obtenu de manière formelle le modèle de LFD à partir de celui de Boltzmann-Fermi-Dirac et Lemou [54] a effectué une analyse spectrale de l'équation de LFD linéarisée. On s'intéresse ici au problème de Cauchy de (1.2.1)-(1.2.3) pour les potentiels durs et maxwellien ($0 \leq \gamma \leq 1$) dans le cas spatialement homogène. A première vue, la borne L^∞ satisfaite par les solutions de (1.2.1) simplifie l'analyse en comparaison avec l'équation de Landau classique où l'on a seulement une borne $L \log L$. En fait, alors que la compacité faible est suffisante pour l'équation de Landau classique, on a besoin ici de compacité forte pour traiter le terme $f(1-f)$. Dans le Chapitre 2, on montre le théorème d'existence et d'unicité suivant :

Théorème 1.2.1 *Soit $\gamma \in [0,1]$. Supposons que f_{in} vérifie (1.2.3) et que $f_{in} \in L^1_{2s_0}(\mathbb{R}^3)$ pour un certain $s_0 > 1$. Alors, il existe une solution faible f de (1.2.1)-(1.2.3) vérifiant, pour tout $t \in \mathbb{R}_+$,*

$$\int_{\mathbb{R}^3} f(t,v)\,dv = \int_{\mathbb{R}^3} f_{in}\,dv, \qquad \int_{\mathbb{R}^3} f(t,v)\,|v|^2\,dv = \int_{\mathbb{R}^3} f_{in}\,|v|^2\,dv$$

et

$$f(1-f) \in L^1_{loc}\big(\mathbb{R}_+; L^1_{2s_0+\gamma}(\mathbb{R}^3)\big); \qquad f \in L^\infty_{loc}\big(\mathbb{R}_+; L^1_{2s_0}(\mathbb{R}^3)\big) \cap L^2_{loc}\big(\mathbb{R}_+; H^1_{2s_0}(\mathbb{R}^3)\big).$$

Si on suppose également que $s_0 \geq 1 + \gamma/2$, alors $t \longmapsto S(f)(t)$ est une fonction croissante et

$$S_{in} := S(f_{in}) \leq S(f)(t) \leq E_{in} + \pi^{3/2} \qquad \text{pour tout } t \in \mathbb{R}_+.$$

De plus, pour $2s_0 > 4\gamma + 11$, cette solution est unique.

Dans l'énoncé de ce théorème, on a utilisé les notations suivantes : pour $s \in \mathbb{R}$, $p \geq 1$ et $k \in \mathbb{N}$, on pose

$$\|f\|^p_{L^p_{2s}} = \int_{\mathbb{R}^3} \left|f(v)\right|^p \left(1+|v|^2\right)^s dv \quad \text{et} \quad \|f\|^2_{H^k_{2s}} = \sum_{0\leq|\alpha|\leq k} \int_{\mathbb{R}^3} \left|\partial^\alpha_v f(v)\right|^2 \left(1+|v|^2\right)^s dv,$$

où $\alpha = (i_1, i_2, i_3) \in \mathbb{N}^3$, $|\alpha| = i_1 + i_2 + i_3$ et $\partial^\alpha_v f = \partial^{i_1}_1 \partial^{i_2}_2 \partial^{i_3}_3 f$.

La preuve de l'existence d'une solution est adaptée des démonstrations effectuées dans [6] et [26] pour l'équation de Landau classique. L'équation de Landau-Fermi-Dirac, tout comme l'équation de Landau classique, présente deux difficultés : ses coefficients sont à la fois non bornés et non locaux. Il s'agit donc de considérer un problème approché dont les coefficients sont bornés mais toujours non locaux. Les solutions de ce problème approché sont obtenues par un argument de point fixe à partir d'un problème aux coefficients bornés et locaux. On prouve des estimations uniformes et on passe ensuite à la limite à l'aide d'un argument de compacité forte. Cette compacité est en fait une conséquence de l'ellipticité uniforme de la matrice $\overline{A}$, que nous énonçons maintenant.

Soient $E_0 > 0$ et $S_0 > 0$. Désignons par $\mathcal{Y}(E_0, S_0)$ l'ensemble des fonctions $f \in L^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ telles que $0 \leq f \leq 1$ p.p. et

$$\int_{\mathbb{R}^3} f(v)\,|v|^2\,dv \leq E_0, \qquad S(f) \geq S_0.$$

Proposition 1.2.2 *Soit $\gamma \in [0,1]$. Soit $f \in \mathcal{Y}(E_0, S_0)$. Alors, il existe une constante $K > 0$, ne dépendant que de γ, E_0 et S_0, telle que*

$$\sum_{i,j} \overline{A}_{i,j}(v)\,\xi_i\,\xi_j \geq K\left(1+|v|^2\right)^{\gamma/2} |\xi|^2, \qquad v \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3.$$

La preuve de cette proposition est basée sur la démonstration de [26] sauf pour la première étape. En effet, pour l'équation de Landau classique, la première étape consiste à minorer $\int_{B_R} f\,dv$ par une quantité strictement positive, ce qui est élémentaire à l'aide de la conservation de la masse et de l'énergie (B_R désigne la boule de centre 0 et de rayon R de $\mathbb{R}^3$). Pour l'équation de LFD, il faut minorer $\int_{B_R} f(1-f)\,dv$ par une quantité strictement positive et l'argument précédent ne suffit plus, le problème étant les points où f est voisin de 1. L'information nécessaire sur l'ensemble $\{v \in \mathbb{R}^3\ ;\ f(v) \sim 1\}$ est contenue dans l'entropie.

En ce qui concerne la preuve de l'unicité, elle suit les mêmes étapes que celle de [26] mais comme $Q_L(f)$ n'est pas quadratique, elle nécessite des résultats d'injection dans des espaces de Sobolev à poids.

Dans le Chapitre 3, on détermine de manière rigoureuse les états d'équilibre de (1.2.1). Pour l'équation de Boltzmann-Fermi-Dirac, Lu [59] a montré que les états d'équilibre sont de deux types : les distributions de Fermi-Dirac et les fonctions caractéristiques de boules de $\mathbb{R}^3$. Comme pour l'équation de Boltzmann-Fermi-Dirac, il devrait y avoir deux classes d'états d'équilibre pour l'équation de Landau-Fermi-Dirac, la classe des distributions de

Fermi-Dirac et une classe d'états d'équilibre dégénérés. Nous allons préciser cette affirmation. Commençons par rappeler la définition d'un état d'équilibre. Les états d'équilibre sont les fonctions qui annulent le terme de dissipation d'entropie (1.2.4), mais il faut pouvoir donner un sens à ce terme. Pour toute fonction mesurable f vérifiant $0 \leq f \leq 1$ p.p., on pose

$$g = \sqrt{f(1-f)} \qquad \text{et} \qquad p(f) = \text{ Arctan }\left(\sqrt{f/(1-f)}\right).$$

Alors $p(f)$ appartient à $L^\infty(\mathbb{R}^3)$ et $\nabla p(f)$ appartient à $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. On écrit alors (1.2.4) sous la forme

$$\int_{\mathbb{R}^3} Q_{LFD}(f)\Big[\ln(1-f) - \ln f\Big]\, dv$$
$$= 2 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left|\Pi(v-v_*)|v-v_*|^{(2+\gamma)/2}\Big[g_*\nabla\big(p(f)\big) - g\nabla_*\big(p(f_*)\big)\Big]\right|^2 dv_*\, dv.$$

On peut ainsi définir la notion d'état d'équilibre de la manière suivante :

Définition 1.2.3 *On dit qu'une fonction $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ est un état d'équilibre de l'équation de LFD si $0 \leq f \leq 1$ p.p. et*

$$\Pi(v-v_*)\left[g_*\, \nabla\big(p(f)\big) - g\, \nabla_*\big(p(f_*)\big)\right] = 0, \qquad \textit{dans } \mathcal{D}'\big(\Omega, \mathbb{R}^3\big), \tag{1.2.5}$$

où

$$\Omega = \{(v, v_*) \in (\mathbb{R}^3)^2; v \neq v_*\}.$$

On remarque tout d'abord que toute fonction $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ qui vérifie

$$0 \leq f \leq 1 \text{ p.p.} \qquad \text{et} \qquad f(1-f) = 0 \text{ p.p.}$$

est solution de (1.2.5). Cela signifie que toute fonction caractéristique d'un ensemble mesurable de mesure finie est solution de (1.2.5). On retrouve ainsi une classe d'équilibres dégénérés comme pour l'équation de Boltzmann-Fermi-Dirac mais cette classe contient strictement celle obtenue pour l'équation de Boltzmann-Fermi-Dirac. On se restreint maintenant aux fonctions qui satisfont (1.2.5) et

$$\text{mes}\big(\{v \in \mathbb{R}^3\ ;\ 0 < f(v) < 1\}\big) \neq 0. \tag{1.2.6}$$

On montre alors le théorème suivant :

Théorème 1.2.4 *Les états d'équilibre de l'équation de LFD vérifiant (1.2.6) sont les distributions de Fermi-Dirac, i.e. les fonctions de la forme :*

$$f(v) = \frac{ae^{-b|v-V_0|^2}}{1 + ae^{-b|v-V_0|^2}},$$

avec $V_0 \in \mathbb{R}^3$ et $a, b > 0$.

La première étape de la preuve consiste à montrer que la fonction f est régulière.

Perspectives

L'équation de Landau-Fermi-Dirac a fait l'objet de si peu de travaux qu'il existe de nombreuses voies de recherche à explorer. Un premier point sera de minorer $(1-f)$ par une constante strictement positive, ce qui permettra d'obtenir, dans le théorème d'existence du Chapitre 2, la même régularité que dans [26]. Le retour vers l'équilibre, l'asymptotique des collisions rasantes et le cas spatialement inhomogène sont également à étudier. Finalement, le théorème d'existence obtenu dans le Chapitre 2 ne concerne que le cas des potentiels durs. Le cas des potentiels mous et plus particulièrement le cas coulombien restent un problème ouvert et on peut espérer que l'existence de l'estimation L^∞ $(0 \leq f \leq 1)$ rende l'analyse plus accessible.

1.2.2 Les systèmes de moments

Cas classique

Jusqu'à maintenant, nous avons considéré une description cinétique d'un système de particules. Les particules sont alors décrites par une fonction de distribution qui dépend du temps $t \in \mathbb{R}_+$, de la position $x \in \mathbb{R}^3$ et de la vitesse $v \in \mathbb{R}^3$. Par conséquent, une simulation numérique d'une équation cinétique nécessite la discrétisation de 7 variables. Elle est donc souvent très coûteuse en terme de temps CPU et de mémoire.

Un second niveau de description couramment utilisé est la description fluide. Les particules sont alors décrites par la densité ρ, la vitesse u et la température T. Ces grandeurs macroscopiques ne dépendent que du temps t et de la position x et satisfont les équations hydrodynamiques (équations d'Euler ou de Navier-Stokes). Cependant, ces équations supposent que le système étudié est proche de l'équilibre thermodynamique, ce qui n'est bien sûr pas toujours réalisé.

Il est donc nécessaire de développer des modèles intermédiaires entre les modèles cinétiques et les modèles fluides. Principalement deux approches sont utilisées : les développements asymptotiques (méthodes de Hilbert et de Chapman-Enskog) et les méthodes de moments.

Les méthodes de Hilbert et de Chapman-Enskog sont utilisées lorsque le libre parcours moyen est petit par rapport à la longueur caractéristique du système. Elles supposent que la fonction de distribution peut être développée autour d'une maxwellienne. Ces deux méthodes supposent que le système est proche de l'équilibre.

Les méthodes de moments sont obtenues à partir du choix d'un espace vectoriel $\mathbb{M}$ de dimension finie de fonctions polynômiales de la vitesse v. Le système d'équations s'obtient en multipliant l'équation de Boltzmann par une base de l'espace $\mathbb{M}$ et en intégrant par rapport à la variable de vitesse. Les inconnues sont les intégrales de la fonction de distribution contre les éléments de la base de $\mathbb{M}$. Par exemple, si $\mathbb{M} = \mathrm{vect}(1, v, |v|^2)$, on obtient les équations locales de conservation de la masse, de la quantité de mouvement et de l'énergie :

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^3} f(t,x,v)\,dv + \nabla_x \cdot \int_{\mathbb{R}^3} f(t,x,v)\,v\,dv &= 0,\\
\partial_t \int_{\mathbb{R}^3} f(t,x,v)\,v\,dv + \nabla_x \cdot \int_{\mathbb{R}^3} f(t,x,v)\,v\otimes v\,dv &= 0,\\
\partial_t \int_{\mathbb{R}^3} f(t,x,v)\,|v|^2\,dv + \nabla_x \cdot \int_{\mathbb{R}^3} f(t,x,v)\,|v|^2\,v\,dv &= 0.
\end{aligned}$$

Les flux et les intégrales de l'opérateur de collision font intervenir des termes qui, en général, ne peuvent pas s'exprimer en fonction des inconnues. Pour fermer le système, un choix de la fonction de distribution est nécessaire. Grad [42, 43] a proposé de fermer le système par le développement en polynômes de Hermite suivant

$$f(t,x,v) = \mathcal{M}(t,x,v)\left(Q_0(t,x)H_0(v) + Q_1(t,x)H_1(v) + Q_3(t,x)H_3(v) + \dots\right),$$

avec

$$\mathcal{M}(t,x,v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}}\, e^{-(v-u(t,x))^2/(2T(t,x))},$$

et

$$Q_k(t,x) = \frac{1}{\rho(t,x)} \int_{\mathbb{R}^3} f(t,x,v)\,H_k(v)\,dv,$$

où les H_i désignent les polynômes de Hermite en dimension 3 orthogonaux par rapport au poids $\mathcal{M}$. Cette approche a quelques inconvénients puisque la positivité de la fonction de distribution n'est pas garantie et les systèmes obtenus ne sont pas tous hyperboliques.

Levermore [55] a formalisé une fermeture basée sur la minimisation de l'entropie. Plus précisément, il s'agit de fermer le système par la fonction f qui minimise l'entropie (1.1.8) sous la contrainte que les moments

$$M_i(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,m_i(v)\,dv, \qquad i = 1,\dots,N, \tag{1.2.7}$$

soient fixés, où $(m_i)_{i=1,\dots,N}$ est une base de $\mathbb{M}$. Formellement, le théorème des extrema liés implique que

$$f(t,x,v) = \exp(\alpha_i(t,x)m_i(v)),$$

où les α_i sont déterminés par la contrainte (1.2.7) et où les conventions usuelles de sommation sont utilisées. Dans ce cas, la fonction de distribution est positive et les systèmes obtenus sont toujours hyperboliques. Cependant, le problème de la réalisabilité des moments se pose. En effet, étant donné un vecteur $M(t,x) \in \mathbb{R}^N$, l'existence d'un vecteur $\alpha(t,x) \in \mathbb{R}^N$ tel que

$$M(t,x) = \int_{\mathbb{R}^3} \exp(\alpha(t,x)\cdot m(v))\,m(v)\,dv, \qquad \text{avec} \quad m = (m_i)_{i=1,\dots,N},$$

n'est pas garantie.

Dans cette stratégie, le choix de l'espace $\mathbb{M}$ doit répondre à certaines conditions. Tout d'abord, on souhaite pouvoir retrouver les équations de la dynamique des fluides. On doit donc avoir

$$1, v, |v|^2 \in \mathbb{M}. \tag{1.2.8}$$

De plus, l'espace $\mathbb{M}$ doit respecter les symétries physiques, c'est à dire l'invariance galiléenne. Etant donnée une fonction f vérifiant l'équation de Boltzmann (1.1.1), on définit, pour tout vecteur $u \in \mathbb{R}^3$ et toute matrice orthogonale O, les actions du groupe galiléen $\mathcal{A}_u f$ et $\mathcal{A}_O f$ sur f par

$$\mathcal{A}_u f(t,x,v) = f(t, x-ut, v-u) \qquad \text{et} \qquad \mathcal{A}_O f(t,x,v) = f(t, O^T x, O^T v),$$

pour tout $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, où O^T désigne la matrice transposée de O. Alors, les fonctions $\mathcal{A}_u f$ et $\mathcal{A}_O f$ sont solutions de l'équation de Boltzmann (1.1.1). L'espace $\mathbb{M}$ doit être compatible avec cette invariance. Cela signifie que si $m \in \mathbb{M}$ alors les fonctions $\mathcal{T}_u m$ et $\mathcal{T}_O m$ définies par

$$\mathcal{T}_u m(v) = m(v-u) \qquad \text{et} \qquad \mathcal{T}_O m(v) = m(O^T v),$$

doivent également appartenir à $\mathbb{M}$. Pour la fermeture utilisée par Levermore, l'espace $\mathbb{M}$ doit satisfaire une dernière condition. On introduit le cône convexe

$$\mathbb{M}_c = \left\{ m \in \mathbb{M} : \int_{\mathbb{R}^3} \exp(m(v))\, dv < \infty \right\}.$$

L'espace $\mathbb{M}$ doit être admissible, c'est à dire que le cône $\mathbb{M}_c$ associé doit avoir un intérieur non vide dans $\mathbb{M}$. L'espace aux 13 moments $(1, v, v\otimes v, |v|^2 v)$ considéré par Grad est un exemple d'espace non admissible. Seuls les polynômes m tels que $m(v) \to -\infty$ quand $|v| \to \infty$ peuvent appartenir à $\mathbb{M}_c$. Par conséquent, seuls les espaces de degré maximal pair peuvent être admissibles. Voici des exemples d'espaces $\mathbb{M}$ vérifiant les trois conditions énoncées ci-dessus

degré maximal 2	$\mathrm{vect}(1, v, \|v\|^2)$,
	$\mathrm{vect}(1, v, v\otimes v)$,
degré maximal 4	$\mathrm{vect}(1, v, v\otimes v, v\,\|v\|^2, \|v\|^4)$,
	$\mathrm{vect}(1, v, v\otimes v, v\otimes v\otimes v, \|v\|^4)$,
	$\mathrm{vect}(1, v, v\otimes v, v\otimes v\otimes v, \|v\|^2 v\otimes v)$,
	$\mathrm{vect}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v)$.

Cas relativiste

Nous nous intéressons ici aux systèmes de moments dans le cas relativiste. Le système aux 14 moments $\mathrm{vect}(1, p, \varepsilon(p), \varepsilon(p)p, p\otimes p)$ a déjà été étudié dans [13, 30, 44, 48, 64]. Dreyer et Weiss [30] se sont également intéressés à la limite classique de ce système. Cependant, la détermination rigoureuse des espaces de moments qui sont compatibles avec la transformation de Lorentz n'a pas encore été considérée. C'est le but du Chapitre 4. Dans le cas relativiste, un espace de moments $\mathbb{M}$ est un espace vectoriel de dimension finie constitué de fonctions polynômiales de la quantité de mouvement p et de l'énergie $\varepsilon(p)$. Comme dans le cas classique, on souhaite que le système obtenu généralise les équations de l'hydrodynamique relativiste. Par conséquent, l'espace $\mathbb{M}$ doit satisfaire

$$1, p, \varepsilon(p) \in \mathbb{M}.$$

En mécanique classique, les changements de référentiels s'effectuent à l'aide des transformations de Galilée. En relativité restreinte, ils s'effectuent grâce aux transformations de Lorentz. Soient $\mathcal{R}$ et $\mathcal{R}'$ deux référentiels tels que $\mathcal{R}$ se déplace à la vitesse $u \in \mathbb{R}^3$, $|u| < c$ par rapport à $\mathcal{R}'$. Notons respectivement (t, x) et (t', x') les coordonnées spatio-temporelles dans $\mathcal{R}$ et $\mathcal{R}'$. Alors, on a $(ct', x') = L_u(ct, x)$ où

$$t' = \gamma_u \left(t + \frac{u \cdot x}{c^2}\right) \qquad \text{et} \qquad x' = x + (\gamma_u - 1)\,(u \cdot x)\,\frac{u}{|u|^2} + \gamma_u\, u\, t, \tag{1.2.9}$$

avec

$$\gamma_u = \left(1 - \frac{|u|^2}{c^2}\right)^{-1/2}.$$

La fonction L_u est appelée transformation propre de Lorentz. Etant donnée une solution f de l'équation de Boltzmann relativiste (1.1.21), on définit, pour tout vecteur $u \in \mathbb{R}^3$, $|u| < c$ et toute matrice orthogonale O, les actions du groupe lorentzien $\mathcal{B}_u f$ et $\mathcal{B}_O f$ sur f par

$$\mathcal{B}_u f(t, x, p) = f(t', x', p') \qquad \text{et} \qquad \mathcal{B}_O f(t, x, p) = f(t, O^T x, O^T p),$$

pour tout $(t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, où t' et x' sont donnés par (1.2.9) et où

$$p' = p + (\gamma_u - 1)\,(u \cdot p)\,\frac{u}{|u|^2} + \gamma_u\,\frac{u\,\varepsilon(p)}{c^2}.$$

Alors, les fonctions $\mathcal{B}_u f$ et $\mathcal{B}_O f$ sont solutions de l'équation de Boltzmann relativiste (1.1.21). L'invariance galiléenne est remplacée ici par l'invariance lorentzienne. Cela signifie que si $m \in \mathbb{M}$ alors les fonctions $\mathcal{T}_u m$ et $\mathcal{T}_O m$ définies par

$$\mathcal{T}_u m(\varepsilon, p) = m(\varepsilon', p') \qquad \text{et} \qquad \mathcal{T}_O m(\varepsilon, p) = m(\varepsilon, O^T p), \quad \text{où} \quad \varepsilon' = \gamma_u(\varepsilon + u \cdot p),$$

doivent également appartenir à $\mathbb{M}$. Comme dans le cas classique, on introduit alors le cône convexe

$$\mathbb{M}_c = \left\{ m \in \mathbb{M} : \int_{\mathbb{R}^3} \exp(m(\varepsilon(p), p))\, dp < \infty \right\},$$

et la notion d'espaces admissibles. Le Chapitre 4 correspond au travail en cours [9]. On y détermine les espaces de moments pouvant être utilisés dans les méthodes aux moments, c'est à dire les espaces $\mathbb{M}$ vérifiant les conditions suivantes :

(I) vect $\big(1,\, p,\, \varepsilon\big) \subset \mathbb{M}$,

(II) $\mathbb{M}$ est invariant par les transformations $\mathcal{T}_u$ et $\mathcal{T}_O$,

(III) le cône $\mathbb{M}_c$ a un intérieur non vide dans $\mathbb{M}$.

On commence par considérer la condition (II). Comme les transformations propres de Lorentz et les rotations dans l'espace des moments forment un groupe de Lie (cf. Annexe B), on utilise la théorie de la représentation des groupes et algèbres de Lie. On obtient alors la proposition suivante :

Proposition 1.2.5 *Pour tout $l \in \mathbb{N}$, notons $\hat{\mathbb{M}}_l$ l'espace vectoriel engendré par les parties réelles et imaginaires de*

$$\sum_{m=\max(q-r,0)}^{q} \frac{(l-r)!\, r!}{(l-r-m)!(r-q+m)!} \binom{q}{m} (\varepsilon/c+p^3)^m(\varepsilon/c-p^3)^{r-q+m}(p^1+ip^2)^{l-r-m}(p^1-ip^2)^{q-m}, \quad (1.2.10)$$

pour $q, r \in [\![0, l]\!]$, $q + r \leq l$. Chaque $\hat{\mathbb{M}}_l$ vérifie la condition (II).

De plus, un sous-espace de dimension finie $\mathbb{M}$ de $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ vérifie la condition (II) si et seulement si il existe $N \in \mathbb{N}$ et des $l_k \in \mathbb{N}$, $k = 1, \ldots, N$ tels que $\mathbb{M}$ est la somme directe des $\hat{\mathbb{M}}_{l_k}$, $k = 1, \ldots, N$.

Ce théorème décrit de manière explicite tous les espaces de dimension finie qui satisfont la condition (II). On obtient les espaces vérifiant les conditions (I) et (II) simplement en ajoutant les moments 1, p et $\varepsilon(p)$ aux espaces précédents. Quant à la condition (III), on vérifie facilement que les espaces obtenus la satisfont. On considère alors les limites classique et ultra-relativiste des systèmes obtenus. Dans la limite classique des espaces de moments relativistes, on obtient des espaces de moments compatibles avec l'invariance galiléenne. Certains des espaces obtenus sont admissibles tandis que d'autres ne le sont pas. Cependant, parmi les espaces admissibles obtenus par cette limite classique, on ne retrouve pas tous les espaces admissibles cités dans la section précédente. Plus précisément, on n'obtient pas les espaces $(1, v, v \otimes v)$, $(1, v, v \otimes v, v\,|v|^2, |v|^4)$ et $(1, v, v \otimes v, v \otimes v \otimes v, |v|^4)$. Comme la mécanique classique est considérée comme une approximation de la mécanique relativiste, un critère possible pour le choix des espaces de moments classiques serait de sélectionner les espaces admissibles qui peuvent être obtenus à partir des systèmes relativistes.

Comme le groupe des rotations en dimension 3 forme également un groupe de Lie, la théorie des représentations s'applique également dans le cas classique. Cependant, une fois que l'on a déterminé les espaces invariants par les rotations, il faut encore tenir compte de l'invariance par les translations. Nous construisons tous ces espaces invariants et nous retrouvons, en particulier, les exemples de systèmes admissibles décrits ci-dessus.

Perspectives

Dans le cas des systèmes de moments relativistes, le problème de la réalisabilité des moments reste ouvert. Seul le cas du système $(1, p, \varepsilon(p))$ a été considéré (cf. [40, 35]). Dans le cas classique, le problème de la réalisabilité des moments a été étudié par Junk [49]. Il a supposé que l'un des moments de la base croît plus vite que les autres à l'infini.

Comme $\varepsilon(p)$ et p sont équivalents à l'infini, cette hypothèse n'est pas pertinente dans le cas relativiste. Notre but, maintenant, est de surmonter ces difficultés et d'obtenir des résultats similaires à ceux de Junk dans le cas relativiste. De plus, pour valider notre approche, une étude numérique de ces systèmes est envisagée.

1.2.3 L'équation de Kac avec thermostat

Les thermostats déterministes

Ainsi que nous l'avons signalé dans la Section 1.1.1, dans un système composé d'un grand nombre de particules identiques, les collisions entre les particules conduisent le système de particules vers l'équilibre s'il n'y a pas de champ de force extérieur. La fonction de distribution converge alors vers une maxwellienne. En revanche, si on impose un champ de force extérieur, il y a création de chaleur et le système est amené hors de l'équilibre. Pour atteindre un état stationnaire il faut retirer l'excès d'énergie. Les thermostats ont été introduits dans ce but en dynamique moléculaire et en physique statistique (cf. [62] et les références citées). Un thermostat est un terme ajouté aux équations du mouvement d'un système soumis à un forçage, de manière à maintenir constante une des variables physiques (l'énergie cinétique, l'énergie interne, le courant, la pression ou l'enthalpie).

On considère un système de N particules soumis à des forces extérieures. Désignons par $X = (x_1, \dots, x_N)$ les coordonnées de ces particules. On a alors

$$\begin{aligned} \partial_t X &= V, \\ \partial_t V &= F_i + F_e - \alpha V, \end{aligned}$$

où les masses des particules sont prises égales à 1, les forces internes entre les particules sont prises en compte dans le terme F_i alors que le forçage extérieur est contenu dans le terme F_e. Le terme de frottement α constitue le thermostat et est déterminé de la manière suivante. Si on désire maintenir l'énergie cinétique $K = V \cdot V/2$ constante, on obtient le thermostat gaussien isocinétique

$$\alpha_c = \frac{(F_i + F_e) \cdot V}{V \cdot V}. \tag{1.2.11}$$

Si on veut fixer l'énergie interne $H = \varphi_i + K$ $(F_i = -\nabla \varphi_i)$, on obtient le thermostat gaussien isoénergétique

$$\alpha_i = \frac{F_e \cdot V}{V \cdot V}.$$

L'adjectif gaussien vient du fait que ces deux thermostats peuvent être obtenus à partir du principe des moindres contraintes de Gauss.

Le thermostat gaussien isocinétique a été utilisé en relation avec le modèle de Lorentz [14, 15, 20, 21] et plus récemment avec l'équation de Kac ainsi que nous allons le voir dans le paragraphe suivant.

Application à l'équation de Kac

On considère, comme dans la Section 1.1.3, un système de N particules ponctuelles qui, à des intervalles de temps exponentiellement distribués, subissent des collisions binaires dont le résultat est donné par (1.1.22)-(1.1.23). Entre les collisions, les particules sont accélérées par un champ de force extérieur constant $E \in \mathbb{R}$. De manière à garder l'énergie cinétique constante, on utilise un thermostat de type (1.2.11). Plus précisément, l'évolution des vitesses $V = (v_1, \ldots, v_N)$ est donnée par

$$\frac{dV}{dt} = \mathcal{E} - \frac{\mathcal{E} \cdot V}{|V|^2} V,$$

où $\mathcal{E} = E(1, \ldots, 1) \in \mathbb{R}^N$.

La densité Ψ_N correspondant à ce système de N particules vérifie alors l'équation

$$\partial_t \Psi_N(t, V) + \nabla \cdot (F \Psi_N) = K(\Psi_N)(t, V),$$

où K est défini comme dans (1.1.24) et où F est donné par

$$F = \mathcal{E} - \frac{\mathcal{E} \cdot V}{|V|^2} V.$$

Sous l'hypothèse de la propagation du chaos et sous certaines conditions de régularité, il a été prouvé dans [74] que la fonction f_N^1 définit par (1.1.25) converge, lorsque $N \to \infty$, vers une solution f d'une équation de Kac modifiée

$$\partial_t f(t, v) + E \partial_v \left((1 - \zeta(t) v) f(t, v) \right) = Q_K(f, f)(t, v), \qquad t \in \mathbb{R}_+, v \in \mathbb{R}, \tag{1.2.12}$$

où $\zeta(t) = \int_{\mathbb{R}} v\, f(t, v)\, dv$ et Q_K est donné, comme précédemment, par

$$Q_K(f, f)(t, v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f(t, v') f(t, v'_*) - f(t, v) f(t, v_*))\, b(\theta)\, d\theta\, dv_*, \tag{1.2.13}$$

avec

$$b(\theta) = \frac{1}{2\pi}. \tag{1.2.14}$$

L'existence de solutions stationnaires pour (1.2.12)-(1.2.14) a été démontrée dans [72]. Le comportement des solutions varie selon la valeur du champ de force extérieur E. Plus précisément, ils ont montré le théorème suivant :

Théorème 1.2.6 *Pour tout $E > 0$, il existe une solution stationnaire f de (1.2.12)-(1.2.14). Pour $E < \sqrt{2}$, $f \in \mathcal{C}(\mathbb{R})$. Pour $E = \sqrt{2}$, f présente une singularité de type logarithmique en $v = \sqrt{2}$. Pour $E > \sqrt{2}$, f présente une singularité de la forme $|v - \kappa|^\gamma$ en $v = \kappa$, où*

$$\kappa = \frac{2E}{\sqrt{1 + 4E^2} - 1} \qquad \text{et} \qquad \gamma = \frac{\kappa}{E} - 1.$$

L'existence de solutions de (1.2.12)-(1.2.14) et leur convergence faible vers un état stationnaire a été étudié dans [73].

Une généralisation naturelle de (1.2.12)-(1.2.14) consiste à remplacer la distribution uniforme b donnée par (1.2.14) par la fonction

$$b(\theta) = |\theta|^{-1-\alpha}, \qquad \theta \in (-\pi, \pi), \qquad \alpha \in (0, 2). \tag{1.2.15}$$

Dans ce cas, b n'est plus intégrable. Desvillettes [25] a montré qu'alors, l'opérateur de collision a un effet régularisant. Pour l'équation de Boltzmann, il a été également prouvé que l'opérateur de collision sans cut-off a un effet régularisant (cf. [3, 27]). Il est intéressant de voir si l'effet régularisant de l'opérateur de collision de (1.2.12)-(1.2.13) avec (1.2.15) est suffisant pour empêcher les fortes valeurs du champ E d'apporter une singularité aux solutions stationnaires. Ceci est l'objet du Chapitre 5, qui est issu du travail en cours [11]. On y prouve le théorème suivant :

Théorème 1.2.7 *Supposons que b vérifie (1.2.15). Pour toutes les valeurs de $E > 0$, il existe une unique solution stationnaire f de (1.2.12)-(1.2.13) telle que les moments de tout ordre de f sont finis et*

$$\int_{\mathbb{R}} f(v)\, dv = 1.$$

De plus, $f \in \mathcal{C}^\infty(\mathbb{R})$.

La preuve du Théorème 1.2.7 suit les étapes suivantes. Dans un premier temps, on montre l'existence d'une solution stationnaire de (1.2.12)-(1.2.13) dans le cas où b est lipschitzienne en adaptant à ce cas les techniques de [72]. L'existence d'une solution stationnaire de (1.2.12)-(1.2.13) dans le cas où b satisfait (1.2.15) s'obtient alors par troncature de b. Pour prouver la régularité des solutions stationnaires de (1.2.12)-(1.2.13), on utilise les techniques de la transformée de Fourier, dans le même esprit que dans [25]. La preuve de l'unicité repose sur le développement en série entière de la transformée de Fourier d'une solution stationnaire f et sur des estimations sur les moments de f. Les résultats théoriques sont illustrés par des résultats numériques.

Perspectives

En ce qui concerne l'équation de Kac sans cut-off en présence d'un thermostat gaussien, seule l'existence et la régularité des solutions stationnaires a été démontrée. L'étape suivante est maintenant d'étudier l'existence de solutions pour l'équation dépendant du temps et la convergence de ces solutions vers les états stationnaires.

1.2.4 L'équation de coagulation de Oort-Hulst-Safronov

L'équation que l'on considère dans cette section et dans le Chapitre 6 n'est pas une équation cinétique mais elle s'en rapproche, de par ses propriétés et de par les techniques utilisées. Elle fait partie des équations de coagulation (cf. [1, 53]). Les processus de coagulation décrivent les phénomènes physiques par lesquels des amas (de particules, gouttelettes, ...) fusionnent pour en former de plus grands. Chaque amas est entièrement déterminé par sa taille ou sa masse. On distingue les modèles discrets où le paramètre de taille appartient à $\mathbb{N}\backslash\{0\}$ des modèles continus où le paramètre de taille appartient à $\mathbb{R}_+ = (0, +\infty)$. Le modèle de coagulation le plus étudié est le modèle de Smoluchowski qui décrit l'évolution de la densité $f(t, x)$ d'amas de taille $x \in \mathbb{R}_+$ à l'instant $t \geq 0$ et s'écrit [67, 68, 63, 29]

$$\partial_t f(t, x) = Q_{SM}(f)(t, x), \qquad t, x \in \mathbb{R}_+,$$

avec

$$Q_{SM}(f)(t, x) = \frac{1}{2} \int_0^x K(x - y, y) f(t, x - y) f(t, y)\, dy - f(t, x) \int_0^\infty K(x, y) f(t, y)\, dy,$$

où K est une fonction symétrique et positive qui décrit le processus physique de coagulation,

$$0 \leq K(x, y) = K(y, x), \qquad (x, y) \in \mathbb{R}_+^2. \tag{1.2.16}$$

Dans ce modèle, le premier terme de $Q_{SM}(f)$ représente la formation d'amas de masse x à partir de deux amas de masses respectives $y \in (0, x)$ et $x - y$, et le second terme de $Q_{SM}(f)$ décrit la coalescence d'un amas de masse x avec un amas de masse y pour former un amas de masse $x + y$. Ainsi, au niveau microscopique, il y a conservation de la masse au cours de chaque réaction de coagulation. Au niveau de f, la masse totale du système à l'instant t est le moment d'ordre 1 de $f(t)$ et on s'attend à ce qu'il reste constant au cours du temps :

$$\int_0^\infty f(t, x)\, x\, dx = \int_0^\infty f(0, x)\, x\, dx, \qquad t \geq 0.$$

Formellement, cette propriété est évidente sur la formulation faible

$$\frac{d}{dt} \int_0^\infty f(t, x)\, \varphi(x)\, dx = \int_0^\infty \int_0^x K(x, y) f(t, x) f(t, y) \left[\varphi(x + y) - \varphi(x) - \varphi(y)\right] dy\, dx,$$

avec le choix $\varphi(x) = x$. Cependant, il se peut qu'en fait, la masse ne reste pas constante au cours du temps (la fonction test $\varphi(x) = x$ n'est alors pas admissible). C'est ce qu'on appelle le phénomène de gélification. Ainsi, pour des noyaux K tels que

$$K(x, y) = x^\alpha y^\beta + x^\beta y^\alpha, \qquad (x, y) \in \mathbb{R}_+^2,$$

avec $0 \leq \alpha \leq \beta \leq 1$ et $\lambda := \alpha + \beta > 1$, si on désigne par f une solution faible de l'équation de Smoluchowski dont la condition initiale f^{in} vérifie

$$f^{in} \in L_1^1(\mathbb{R}_+) = L^1(\mathbb{R}_+, (1 + x)dx), \qquad f_{in} \not\equiv 0, \qquad \text{et} \qquad f^{in} \geq 0 \quad p.p.,$$

il a été démontré [33] que l'on a $T_{gel} < +\infty$, où

$$T_{gel} = \inf \left\{ t \geq 0, \quad \int_0^\infty f(t, x)\, x\, dx < \int_0^\infty f^{in}(x)\, x\, dx \right\}.$$

Cette perte de masse s'explique par le fait qu'une partie de la masse part à l'infini, ce qui, d'un point de vue physique, signifie qu'il y a création, en temps fini, de particules de masse infinie.

Pour décrire le processus d'agrégation d'objets protoplanétaires, Oort, van de Hulst [65] et Safronov [66] ont proposé le modèle suivant

$$\partial_t f(t,x) = Q_{OHS}(f)(t,x), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}_+, \tag{1.2.17}$$

avec

$$Q_{OHS}(f)(t,x) = -\partial_x \left[f(t,x) \int_0^x y\, K(x,y) f(t,y)\, dy \right] - \int_x^\infty K(x,y) f(t,x) f(t,y)\, dy, \tag{1.2.18}$$

où, comme précédemment, la fonction K vérifie (1.2.16). L'équation (1.2.17) est complétée par la condition initiale

$$f^{in} \in L^1_1(\mathbb{R}_+) = L^1(\mathbb{R}_+, (1+x)dx) \qquad \text{et} \qquad f^{in} \geq 0 \quad p.p. \tag{1.2.19}$$

En multipliant (1.2.17) par une fonction test φ, on obtient, avec (1.2.16), la formulation faible suivante

$$\frac{d}{dt} \int_0^\infty f(t,x)\, \varphi(x)\, dx = \int_0^\infty \int_0^x K(x,y) f(t,x) f(t,y) \left[y\, \partial_x \varphi(x) - \varphi(y) \right] dy\, dx. \tag{1.2.20}$$

D'une part, le modèle de OHS peut être formellement relié au modèle de Smoluchowski puisque pour $x \gg y$, on a

$$\varphi(x+y) - \varphi(x) - \varphi(y) \sim y\, \partial_x \varphi(x) - \varphi(y).$$

Par conséquent, l'équation de OHS peut être formellement obtenue à partir de l'équation de Smoluchowski. Le lecteur pourra consulter [50] pour un énoncé plus précis de ce résultat. D'autre part, l'équation de Oort-Hulst-Safronov (OHS) vérifie le même type de propriétés que l'équation de Smoluchowski, sauf la propagation à vitesse finie.

Si ∂_x est approché par des différences finies et si les intégrales sont approchées par des sommes de Riemann, on obtient la version discrète suivante de (1.2.17)

$$\frac{dc_i}{dt} = Q_i(c) \qquad \text{dans} \quad (0,+\infty), \tag{1.2.21}$$

pour $i \geq 1$, où $c = (c_i)_{i \geq 1}$,

$$Q_i(c) = c_{i-1} \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j - c_i \sum_{j=1}^{i} j\, K_{i,j}\, c_j - \sum_{j=i}^{\infty} K_{i,j}\, c_i\, c_j, \tag{1.2.22}$$

et $K_{i,j} = K_{j,i} \geq 0$ est le noyau de coagulation discret. Cette équation sera appelée équation de OHS discrète (dOHS). Les équations (1.2.21)-(1.2.22) représentent un cas particulier d'une famille à deux paramètres de modèles de coagulation discrets introduite par Dubovski [31] où Dubovski a mis en évidence un lien entre les équations de OHS et de dOHS. Dans le Chapitre 6, on précise ce lien. Ce chapitre a été l'objet de la publication [8]. On y montre que, pour un choix convenable de noyaux discrets pour (1.2.21), les solutions de ces équations convergent vers une solution de l'équation de OHS. Une relation similaire a déjà été établie dans [52] entre les équations classiques continue et discrète de coagulation-fragmentation de Smoluchowski.

Plus précisément, on suppose que le noyau K vérifie, pour chaque $R \geq 1$, les hypothèses suivantes

$$K \in W^{1,\infty}_{loc}([0,+\infty)^2), \tag{1.2.23}$$

$$\partial_x K(x,y) \geq -\alpha, \qquad \text{pour un certain } \alpha \geq 0, \tag{1.2.24}$$

$$\omega_R(y) = \sup_{x\in[0,R]} \frac{K(x,y)}{y} \longrightarrow 0 \quad \text{quand} \quad y \to +\infty. \tag{1.2.25}$$

Sous de telles hypothèses, il a déjà été démontré, dans [50], l'existence d'une solution faible de (1.2.17)-(1.2.19). Soit $\varepsilon \in (0,1)$. On pose

$$\Lambda_i^\varepsilon = [(i-1/2)\varepsilon, (i+1/2)\varepsilon) \qquad \text{et} \qquad \chi_i^\varepsilon = \mathbf{1}_{\Lambda_i^\varepsilon},$$

pour $i \geq 1$. On définit ensuite les coefficients discrets soit par

$$K_{i,j}^\varepsilon = \frac{1}{\varepsilon} \int_{\Lambda_i^\varepsilon \times \Lambda_j^\varepsilon} K(x,y)\, dy\, dx, \tag{1.2.26}$$

ou par

$$K_{i,j}^\varepsilon = \varepsilon K(\varepsilon i, \varepsilon j), \tag{1.2.27}$$

pour $i, j \geq 1$. La condition initiale discrète $c^{in,\varepsilon} = (c_i^{in,\varepsilon})_{i\geq 1}$ est alors donnée par

$$c_i^{in,\varepsilon} = \frac{1}{\varepsilon} \int_{\Lambda_i^\varepsilon} f^{in}(x)\, dx, \qquad i \geq 1. \tag{1.2.28}$$

Soit $c^\varepsilon = (c_i^\varepsilon)_{i\geq 1}$ une solution de l'équation dOHS avec les coefficients $K_{i,j}^\varepsilon$ et la condition initiale $c^{in,\varepsilon}$. On introduit, pour $(t,x) \in \mathbb{R}_+^2$,

$$f_\varepsilon(t,x) = \sum_{i=1}^\infty c_i^\varepsilon(t)\, \chi_i^\varepsilon(x), \tag{1.2.29}$$

et on montre le théorème suivant :

Théorème 1.2.8 *Supposons que K vérifie (1.2.16), (1.2.23)-(1.2.25) et que la condition initiale f^{in} vérifie (1.2.19). On note c^ε une solution de d'équation dOHS (1.2.21) avec les coefficient $K_{i,j}^\varepsilon$ définis par (1.2.26) ou (1.2.27) et la donnée initiale $c^{in,\varepsilon}$ donnée par (1.2.28) telle que l'on ait*

$$\sum_{i=1}^\infty i\, c_i^\varepsilon(t) \leq \sum_{i=1}^\infty i\, c_i^{in,\varepsilon}, \qquad t \geq 0. \tag{1.2.30}$$

Soit f_ε la fonction définie par (1.2.29). Alors, il existe une solution faible $f \in \mathcal{C}([0,T]; w-L^1(\mathbb{R}_+)) \cap L^\infty(0,T; L^1_1(\mathbb{R}_+))$ de l'équation OHS (1.2.17) de condition initiale f^{in} et une sous-suite (f_{ε_n}) de (f_ε) telle que

$$f_{\varepsilon_n} \longrightarrow f \qquad \text{dans } \mathcal{C}([0,T]; w - L^1(\mathbb{R}_+)) \qquad \text{pour chaque } T \in \mathbb{R}_+.$$

La preuve de ce théorème repose sur le fait que l'équation dOHS peut être vue comme une équation de OHS modifiée. La première étape consiste à montrer des estimations uniformes en ε pour les fonctions f_ε, ce qui garantit que les f_ε appartiennent à un ensemble faiblement compact de L^1. On passe alors à la limite $\varepsilon \to 0$.

Le Théorème 1.2.8 utilise des solutions de l'équation dOHS. Par conséquent, pour avoir une démonstration complète, il nous reste à montrer l'existence de ces solutions. C'est l'objet de la proposition suivante.

Proposition 1.2.9 *Soit $(K_{i,j})$ une suite de nombres réels positifs tels que*

$$K_{i,j} = K_{j,i} \geq 0 \qquad \text{et} \qquad \lim_{k \to +\infty} \frac{K_{i,k}}{k} = 0, \qquad i,j \geq 1.$$

Si $c^{in} = (c_i^{in})_{i \geq 1}$ est une suite de nombres réels positifs telle que

$$\sum_{i=1}^{\infty} i\, c_i^{in} < +\infty,$$

alors, il existe au moins une solution c de l'équation dOHS (1.2.21) de condition initiale c^{in} sur $[0,+\infty)$ telle que $c_i \in \mathcal{C}([0,+\infty))$ pour tout $i \geq 1$ et telle que (1.2.30) soit vérifié.

Pour prouver cette proposition, on procède comme dans [12, 69]. Tout d'abord, on approche l'équation dOHS par un système d'équations différentielles ordinaires. On passe alors à la limite grâce au théorème d'Arzela-Ascoli.

D'après [50, Theorem 2.6], si la donnée initiale f^{in} est à support compact dans $[0,+\infty)$ alors, la solution $f(t)$ de OHS reste à support compact pour $t \in [0, T_*)$, où T_* peut être fini ou infini selon le noyau de coagulation K. Au contraire, pour l'équation dOHS, on a le résultat suivant

Proposition 1.2.10 *Supposons que $K_{i,i} > 0$ pour $i \geq 1$. Soit $c^{in} = (c_i^{in})_{i\geq 1}$ une suite de nombres réels positifs telle que $c_k^{in} > 0$ pour un certain $k \geq 1$. Soit $c = (c_i)_{i \geq 1}$ une solution de l'équation dOHS (1.2.21) sur un intervalle $[0,T)$ de condition initiale c^{in}. Alors, pour tout $i \geq k$ et $t \in (0,T)$, $c_i(t) > 0$.*

L'approximation de l'équation OHS ci-dessus peut être utilisée pour des simulations numériques. Dans la Section 6.5, la convergence du Théorème 1.2.8 est illustrée par une comparaison numérique entre une solution exacte et la solution discrète associée.

Perspectives

Une question intéressante est le comportement au voisinage de T_{gel} des solutions de l'équation de OHS. Pour l'équation de coagulation de Smoluchowski, les physiciens estiment que la fonction de distribution devrait se comporter comme une solution auto-similaire au voisinage de T_{gel} et on s'attend au même type de comportement pour l'équation de OHS. Pour vérifier cette conjecture, un premier point consiste à étudier l'existence de solutions auto-similaires. Une telle étude a déjà été effectuée pour l'équation de Smoluchowski [34, 37].

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Partie I

Equation de Landau-Fermi-Dirac

Cette partie concerne le modèle de Landau-Fermi-Dirac introduit dans la Section 1.2.1. Elle est composée de deux chapitres et d'une annexe. Le premier chapitre traite de l'existence de solutions pour cette équation dans le cas spatialement homogène. Il a fait l'objet d'un article publié dans le journal *Proceedings of the Royal Society of Edinburgh Section A*. Ensuite, dans l'annexe, on donne une démonstration détaillée d'un théorème du chapitre précédent. Dans le second chapitre, nous déterminons les états d'équilibre de ce modèle. Ce second travail a fait l'objet d'une publication dans les actes du colloque "Nonlocal elliptic and parabolic problems" parus dans les *Banach Center Publications.*

Chapitre 2

Well-posedness for the spatially homogeneous Landau-Fermi-Dirac equation for hard potentials

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Abstract

We study the Cauchy problem for the spatially homogeneous Landau equation for Fermi-Dirac particles, in the case of hard and Maxwellian potentials. We establish existence and uniqueness of a weak solution for a large class of initial data.

2.1 Introduction

Kinetic theory aims at modelling a gas or a plasma when one is interested rather in the statistical properties of the gas than in the state of each gas particle. The evolution of the gas is then described by a distribution function $f = f(t, x, v) \geq 0$ which represents the (local) density of particles with velocity $v \in \mathbb{R}^3$ at position $x \in \mathbb{R}^3$ and time $t \in \mathbb{R}_+ := [0, +\infty[$.

In the absence of interactions (or collisions) between particles, the evolution of f is given by the free transport equation. When the effect of collisions is included, f satisfies the celebrated Boltzmann equation or related models [3, 4, 5, 20]. In particular, while the Boltzmann equation is valid for neutral particles or weakly ionized plasmas, the modelling of completely ionized plasmas introduces a new model, the Landau equation, which is obtained as a limit of the Boltzmann equation when grazing collisions prevail (cf. [5, 8, 9, 20]). Also quantum effects such as the Pauli exclusion principle should sometimes be taken into account and both the Boltzmann and Landau equations have to be modified accordingly in that case [5, 7, 20]. We also mention that a Landau equation with Fermi statistics arises in the modelling of self gravitating particles [6, 16].

In this paper, we study a modified Landau equation accounting for the Pauli exclusion principle which reads:

$$\partial_t f + v \cdot \nabla_x f = Q_L(f),$$

where

$$Q_L(f) = \nabla_v \cdot \int \Psi(v - v_*)\, \Pi(v - v_*) \Big\{ f_*(1 - \delta f_*) \nabla f - f(1 - \delta f) \nabla f_* \Big\}\, dv_*,$$

with $\delta = 1$, $f = f(t, v)$, $f_* = f(t, v_*)$, $\Pi(z)$ denotes the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}, \qquad 1 \leq i, j \leq 3,$$

and Ψ is a function such as $\Psi(z) = |z|^{2+\gamma}$, $-3 \leq \gamma \leq 1$. The choice $\Psi(z) = |z|^{2+\gamma}$ corresponds to inverse power law potentials, among which we distinguish the Coulomb potential ($\gamma = -3$), soft potentials ($-3 < \gamma < 0$), the Maxwellian potential ($\gamma = 0$) and hard potentials ($0 < \gamma \leq 1$). We recall here that the Coulomb potential is however the only one to have a physical relevance.

Taking $\delta = 0$ in $Q_L(f)$ corresponds to the classical Landau equation, while the Landau-Fermi-Dirac (LFD) equation and the Landau-Bose-Einstein (LBE) equation correspond to $\delta = 1$ and $\delta = -1$, respectively. Only the case $\delta = 1$ will be considered herein and our aim is to investigate the existence and uniqueness of weak solutions to the LFD equation in a spatially homogeneous setting, that is $f = f(t, v)$ and satisfies

$$\partial_t f = Q_L(f), \tag{2.1.1}$$

with $\delta = 1$. We point out that the Pauli exclusion principle implies that a solution to (2.1.1) must satisfy $0 \leq f \leq 1$.

While the classical Boltzmann and Landau equations have been the subject of several papers (see [3, 4, 11, 30] for the Boltzmann equation and [2, 10, 15, 29] for the Landau equation, and the references therein), fewer studies have been devoted to the Boltzmann-Fermi-Dirac (BFD) equation and to the LFD equation. Concerning the former, the spatially inhomogeneous Cauchy problem has been studied in [1, 12, 22] for cross sections satisfying Grad's cut-off assumption. In a spatially homogeneous setting, existence of solutions to the BFD equation is investigated in [13, 24] for more realistic cross sections, and their large time behaviour as well [13, 24, 25]. To our knowledge, the problem of existence and uniqueness of solutions to the LFD equation has not been yet considered, and the only works on this model concern a formal derivation from the BFD equation in the grazing collisions limit [7] and a spectral analysis of its linearization near an equilibrium [18]. Therefore, our purpose is to investigate the well-posedness of the Cauchy problem for the LFD equation in a spatially homogeneous setting for hard or Maxwellian potentials. As already mentioned, the Pauli exclusion principle implies that solutions to the LFD equation should satisfy the L^∞-bound $0 \le f \le 1$. On the one hand, this L^∞-bound simplifies the analysis in comparison to the classical Landau equation where only a bound in $L \log L$ is available. On the other hand, the term $f(1-\delta f)$ is nonlinear for $\delta = 1$ and requires strong compactness arguments to be handled (weak compactness is sufficient for the classical Landau equation where $\delta = 0$, since the term $f(1-\delta f) = f$ is linear in that case).

We now describe the contents of the paper. We set notations and state our main results in the next section: existence, propagation of moments, uniqueness (Theorem 2.2.2), ellipticity of $Q_L(f)$ (Proposition 2.2.3). *A priori* estimates are gathered in Section 2.3 and are used in Section 2.4 to prove the existence of a solution to the LFD equation. Finally, the uniqueness result stated in Theorem 2.2.2 is proved in Section 2.5.

2.2 Main results

We first introduce some notations and definitions. For $s \in \mathbb{R}$, $p \ge 1$ and $k \in \mathbb{N}$, we set

$$\begin{aligned}
L^p_{2s}(\mathbb{R}^3) &:= L^p\big(\mathbb{R}^3; \big(1+|v|^2\big)^s\, dv\big), \\
\|f\|^p_{L^p_{2s}} &= \int \big|f(v)\big|^p \big(1+|v|^2\big)^s\, dv, \\
\|f\|^2_{H^k_{2s}} &= \sum_{0 \le |\alpha| \le k} \int \big|\partial^\alpha_v f(v)\big|^2 \big(1+|v|^2\big)^s\, dv,
\end{aligned}$$

where $\alpha = (i_1, i_2, i_3) \in \mathbb{N}^3$, $|\alpha| = i_1 + i_2 + i_3$ and $\partial^\alpha_v f = \partial_1^{i_1} \partial_2^{i_2} \partial_3^{i_3} f$.

For $s \ge 0$ and $f \in L^1_{2s}(\mathbb{R}^3)$, we denote by $M_{2s}(f)$ the moment of order $2s$ of f, that is

$$M_{2s}(f) = \int \big|f(v)\big|\, |v|^{2s}\, dv.$$

For $(i,j) \in [\![1,3]\!]^2$, we define

$$a(z) = (a_{i,j}(z))_{i,j} \quad \text{with} \quad a_{i,j}(z) = |z|^{\gamma+2}\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right),$$
$$b_i(z) = \sum_k \partial_k a_{i,k}(z) = -2\, z_i\, |z|^\gamma,$$
$$c(z) = \sum_{k,l} \partial^2_{kl} a_{k,l}(z) = -2\,(\gamma+3)\,|z|^\gamma,$$

and, when no confusion can occur, we write $\overline{A} = (\overline{A}_{i,j})$, $\bar{b} = (\bar{b}_i)$, $\overline{B} = (\overline{B}_i)$ with

$$\bar{b}_i = b_i * f, \qquad \bar{c} = c * f,$$
$$\overline{A}_{i,j} = a_{i,j} * \big(f(1-f)\big), \qquad \overline{B}_i = b_i * \big(f(1-f)\big).$$

Otherwise, we use the notations $\overline{A}^f_{i,j}$, $\bar{b}^f_i$, $\overline{B}^f_i$, $\bar{c}^f$ instead of $\overline{A}_{i,j}$, $\bar{b}_i$, $\overline{B}_i$ and $\bar{c}$.

With these notations, the LFD equation can then be written alternatively under the form

$$\partial_t f = \nabla \cdot \big(\overline{A}\, \nabla f - \bar{b}\, f(1-f)\big), \tag{2.2.1}$$

or

$$\partial_t f = \sum_{i,j} \overline{A}_{i,j}\, \partial^2_{i,j} f + \big(\overline{B} - \bar{b}\,(1-2f)\big) \cdot \nabla f - \bar{c}\, f(1-f),$$

and is supplemented with the initial datum

$$f(0) = f_{in}, \tag{2.2.2}$$

where

$$f_{in} \in L^1_2(\mathbb{R}^3), \quad 0 \le f_{in} \le 1 \ \text{ a.e.} \quad \text{and} \quad f_{in} \not\equiv 0. \tag{2.2.3}$$

We note that the last assumption is not restrictive since when $f_{in} \equiv 0$, $f \equiv 0$ is a solution to (2.2.1), (2.2.2).

The usual *a priori* estimates are here available. Indeed, one can formally verify that solutions preserve mass and energy, namely, for every $t \ge 0$

$$M_0(f)(t) = \int f(t,v)\, dv = \int f_{in}\, dv := M_{in}, \tag{2.2.4}$$
$$M_2(f)(t) = \int f(t,v)\, |v|^2\, dv = \int f_{in}\, |v|^2\, dv := E_{in}. \tag{2.2.5}$$

Moreover, introducing the entropy $S(f)$ for Fermi-Dirac particles defined by

$$S(f) = -\int \Big[f \ln f + (1-f)\ln(1-f)\Big]\, dv \ge 0,$$

one can see, still formally, that $t \longmapsto S(f)(t)$ is a non-decreasing function.

Definition 2.2.1 *Consider f_{in} satisfying (2.2.3). A weak solution to the LFD equation (2.2.1), (2.2.2) is a function f satisfying*

$$\begin{aligned}
&(i) \quad f \in L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^3)) \cap \mathcal{C}(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^3)), f(1-f) \in L^1_{loc}(\mathbb{R}_+; L^1_{2+\gamma}(\mathbb{R}^3));\\
&(ii) \quad 0 \le f \le 1 \text{ and } f(0) = f_{in};\\
&(iii) \quad \forall t \ge 0, \quad \int f(t,v)\,|v|^2\,dv \le \int f_{in}(v)\,|v|^2\,dv;\\
&(iv) \quad \forall \varphi \in \mathcal{D}\big(\mathbb{R}^3\big), \forall s,t \ge 0;
\end{aligned}$$

$$\int f(t,v)\varphi(v)\,dv - \int f(s,v)\varphi(v)\,dv$$
$$= \int_s^t d\tau \left[\sum_{i,j} \int \overline{A}_{i,j}\, f\, \partial^2_{i,j}\varphi\,dv + \int f\,\overline{B}\cdot\nabla\varphi\,dv + \int f(1-f)\,\overline{b}\cdot\nabla\varphi\,dv \right].$$

Our main result is the following.

Theorem 2.2.2 *Consider f_{in} satisfying (2.2.3) and assume further that $f_{in} \in L^1_{2s_0}(\mathbb{R}^3)$ for some $s_0 > 1$. Then, there exists a weak solution f to (2.2.1), (2.2.2) satisfying (2.2.4), (2.2.5) and*

$$f(1-f) \in L^1_{loc}\big(\mathbb{R}_+; L^1_{2s_0+\gamma}\big(\mathbb{R}^3\big)\big); \qquad f \in L^\infty_{loc}\big(\mathbb{R}_+; L^1_{2s_0}\big(\mathbb{R}^3\big)\big) \cap L^2_{loc}\big(\mathbb{R}_+; H^1_{2s_0}\big(\mathbb{R}^3\big)\big).$$

If we also suppose that $s_0 \ge 1 + \gamma/2$, $t \longmapsto S(f)(t)$ is a non-decreasing function and

$$S_{in} := S(f_{in}) \le S(f)(t) \le E_{in} + \pi^{3/2} \qquad \text{for every } t \in \mathbb{R}_+.$$

Moreover, for $2s_0 > 4\gamma + 11$, such a solution is unique.

The existence proof is adapted from that of [2, 10] and is performed in three steps: analysis of a regularized equation, uniform estimates and passage to the limit by a compactness argument. At this stage, we recall that, owing to the cubic nature of $Q_L(f)$, a weak compactness argument is not sufficient. Strong compactness is actually a consequence of the uniform ellipticity of the matrix $\overline{A}$ which we state now.

We fix $E_0 > 0$ and $S_0 > 0$ and denote by $\mathcal{Y}(E_0, S_0)$ the set of functions $f \in L^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that $0 \le f \le 1$ a.e. and

$$M_2(f) \le E_0, \qquad S(f) \ge S_0.$$

Proposition 2.2.3 *Let $f \in \mathcal{Y}(E_0, S_0)$. Then there exists a constant $K > 0$, depending only on γ, E_0 and S_0, such that*

$$\sum_{i,j} \overline{A}_{i,j}(v)\,\xi_i\,\xi_j \ge K\big(1+|v|^2\big)^{\gamma/2}\,|\xi|^2, \qquad v \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3.$$

As for the uniqueness proof, it follows the lines from that of [10], but the non-quadratic nature of $Q_L(f)$ requires the use of an embedding lemma for weighted Sobolev spaces.

2.3 *A priori* estimates

2.3.1 Uniform ellipticity

We first prove Proposition 2.2.3, and proceed as in [10, Proposition 4] for the Landau equation with some modifications. Indeed, for the classical Landau equation, the first step is a positive bound from below of $\|f\|_{L^1(B_R)}$ which is straightforward by (2.2.4) and (2.2.5) (B_R denotes the ball with centre 0 and radius R). For the LFD equation, we need a positive bound from below of $\|f(1-f)\|_{L^1(B_R)}$ and we realize that the arguments of [10, Proposition 4] provide no information for velocities where f is close to 1. However, for such velocities, the needed information are to be found in the entropy.

Lemma 2.3.1 *There exist constants $\eta_* > 0$ and $R_* \geq 1$, depending only on E_0 and S_0, such that*

$$\int_{B_{R_*}} f(1-f)\,dv \geq \eta_* > 0 \qquad \text{for every } f \in \mathcal{Y}(E_0, S_0).$$

Proof. Let $f \in \mathcal{Y}(E_0, S_0)$. For every $R \geq 1$, we have

$$S_0 \leq \int_{B_R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\,dv + \int_{|v|\geq R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\,dv. \quad (2.3.1)$$

Step 1. We first consider the integral over B_R. Let $\varepsilon, \alpha \in (0,1)$. Since

$$|\ln r| \leq (1-r)/\varepsilon \quad \text{if} \quad r \in (\varepsilon, 1), \quad (2.3.2)$$

$$|\ln(1-r)| \leq r/\varepsilon \quad \text{if} \quad r \in (0, 1-\varepsilon), \quad (2.3.3)$$

we deduce that

$$\begin{aligned}\int_{B_R} f\,|\ln f|\,dv &\leq \int_{B_R\cap\{f\geq\varepsilon\}} f\,|\ln f|\,dv + \int_{B_R\cap\{f\leq\varepsilon\}} f\,|\ln f|\,dv \\ &\leq \frac{1}{\varepsilon}\int_{B_R} f(1-f)\,dv + \varepsilon^{\alpha}\int_{B_R} f^{1-\alpha}\,|\ln f|\,dv,\end{aligned}$$

and, similarly,

$$\int_{B_R} (1-f)\,|\ln(1-f)|\,dv \leq \frac{1}{\varepsilon}\int_{B_R} f(1-f)\,dv + \varepsilon^{\alpha}\int_{B_R} (1-f)^{1-\alpha}\,|\ln(1-f)|\,dv.$$

As $r \longmapsto r^{1-\alpha}\,|\ln r|$ is bounded on $[0,1]$, we obtain, choosing $\varepsilon = R^{-4/\alpha}$,

$$\int_{B_R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\,dv \;\leq\; 2R^{4/\alpha}\int_{B_R} f(1-f)\,dv + \frac{C_1(\alpha)}{R}\,. \quad (2.3.4)$$

Step 2. It remains now to consider the second integral of (2.3.1). On the one hand, thanks to the Hölder inequality and the boundedness of $r \longmapsto r^\alpha\,|\ln r|$ on $[0,1]$, we obtain

$$\begin{aligned}\int_{|v|\geq R} f\,|\ln f|\,dv &= \int_{|v|\geq R} f^{1-\alpha}\,|v|^{2(1-\alpha)}\,\frac{f^\alpha\,|\ln f|}{|v|^{2(1-\alpha)}}\,dv\\ &\leq C_2(\alpha)\left(\int_{|v|\geq R} f\,|v|^2\,dv\right)^{1-\alpha}\left(\int_{|v|\geq R}|v|^{-2(1-\alpha)/\alpha}\,dv\right)^{\alpha}.\end{aligned}$$

We fix $\alpha = 1/5$ and conclude that

$$\int_{|v|\geq R} f\,|\ln f|\,dv \leq C\,\frac{E_0^{1-\alpha}}{R}. \tag{2.3.5}$$

On the other hand, using (2.3.3) with $\varepsilon = 1/2$ leads to

$$\begin{aligned}\int_{|v|\geq R}(1-f)\,|\ln(1-f)|\,dv &\leq 2\int_{\{|v|\geq R\}\cap\{f\leq 1/2\}} f(1-f)\,dv + \frac{1}{e}\int_{\{|v|\geq R\}\cap\{f\geq 1/2\}} f\,dv\\ &\leq \frac{2}{R^2}\int f\,|v|^2\,dv + \frac{1}{eR^2}\int f\,|v|^2\,dv.\end{aligned}$$

Hence, for $R \geq 1$,

$$\int_{|v|\geq R}(1-f)\,|\ln(1-f)|\,dv \leq \frac{3E_0}{R^2} \leq \frac{3E_0}{R}. \tag{2.3.6}$$

From (2.3.5) and (2.3.6), we deduce

$$\int_{|v|\geq R}\big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\,dv \leq \frac{C_3(E_0)}{R}. \tag{2.3.7}$$

Step 3. Substituting the inequalities (2.3.4) and (2.3.7) into (2.3.1) gives

$$S_0 - \frac{C_1(1/5) + C_3(E_0)}{R} \leq 2R^{20}\int_{B_R} f(1-f)\,dv.$$

The choice

$$R_* = 2\,\frac{C_1(1/5) + C_3(E_0)}{S_0},$$

then completes the proof of Lemma 2.3.1. □

Proof of Proposition 2.2.3. Owing to Lemma 2.3.1, the remainder of the proof of Proposition 2.2.3 is similar to that of [10, Proposition 4], to which we refer. □

2.3.2 Propagation of moments

We now show (formally) the propagation of moments for solutions to the LFD equation (2.2.1), (2.2.2), which, in turn, implies an H^1-estimate (still formally). All the computations we perform here will be justified in Section 2.4.2 by means of smooth approximating solutions.

Let f be a smooth solution to (2.2.1), (2.2.2). Multiplying (2.2.1) by 1 and $|v|^2$ and integrating with respect to v lead, after some integrations by parts, to the conservation of mass (2.2.4) and energy (2.2.5). Also, after multiplying (2.2.1) by $\ln f - \ln(1-f)$ and integrating over $\mathbb{R}^3$, the non-negativity of the matrix a ensures that the entropy $S(f)$ is a non-decreasing function of time. From now on, C_i, $i \geq 1$ denote positive constants depending only on γ, M_{in}, E_{in} and S_{in}. The dependence of the C_i's upon additional parameters will be indicated explicitly.

Lemma 2.3.2 *Assume that $f_{in} \in L^1_{2s}(\mathbb{R}^3)$ for some $s > 1$. Then, for every $T > 0$, there exists a constant $\Gamma(s, T, \|f_{in}\|_{L^1_{2s}})$ depending only on s, T and $\|f_{in}\|_{L^1_{2s}}$ such that*

$$\sup_{t\in[0,T]} \|f(t)\|_{L^1_{2s}} + \int_0^T \iint |v-v_*|^\gamma \, |v_*|^{2s} \, f f_*(1-f_*) \, dv \, dv_* \, d\tau \leq \Gamma(s, T, \|f_{in}\|_{L^1_{2s}}).$$

In particular, $f(1-f) \in L^1_{loc}(\mathbb{R}_+; L^1_{2s+\gamma}(\mathbb{R}^3))$.

Proof. Let φ be a smooth function on $\mathbb{R}^3$ and multiply (2.2.1) by φ. After integrating over $\mathbb{R}^3$ and some integrations by parts, we obtain:

$$\frac{d}{dt}\int f(t,v)\,\varphi(v)\,dv = \sum_{i,j} \iint f f_*(1-f_*)\, a_{i,j}(v-v_*)\, \partial^2_{i,j}\varphi \, dv\, dv_* + \iint f f_* \, [2 - f - f_*] \; b(v-v_*)\cdot\nabla\varphi\, dv\, dv_*. \quad (2.3.8)$$

We take $\varphi(v) = \Phi\big(|v|^2\big)$ in (2.3.8), where Φ is a convex function. As

$$\begin{aligned} \sum_i a_{i,i}(v-v_*) &= 2\,|v-v_*|^{\gamma+2}, \\ \sum_{i,j} a_{i,j}(v-v_*)\, v_i v_j &= |v-v_*|^\gamma \left(|v|^2|v_*|^2 - (v\cdot v_*)^2\right), \\ \sum_j b_j(v-v_*)\, v_j &= -2\,|v-v_*|^\gamma \left(|v|^2 - (v\cdot v_*)\right), \end{aligned}$$

formula (2.3.8) becomes

$$\frac{d}{dt}\int f(t,v)\,\Phi\big(|v|^2\big)\,dv = 4\iint f f_*(1-f_*)\,|v-v_*|^\gamma\,\Lambda^\Phi(v,v_*)\,dv\,dv_*,$$

where

$$\Lambda^\Phi(v,v_*) = \big(|v_*|^2 - (v\cdot v_*)\big)\big(\Phi'\big(|v|^2\big) - \Phi'\big(|v_*|^2\big)\big) + \big(|v|^2|v_*|^2 - (v\cdot v_*)^2\big)\Phi''\big(|v|^2\big).$$

Since Φ is convex, Φ'' is non-negative and, consequently,

$$\Lambda^\Phi(v,v_*) \le \big(|v_*|^2 - (v\cdot v_*)\big)\big(\Phi'\big(|v|^2\big) - \Phi'\big(|v_*|^2\big)\big) + |v|^2|v_*|^2\Phi''\big(|v|^2\big).$$

Let $\Phi(r) = r^s$, $s>1$. Since $(v\cdot v_*) \le |v||v_*|$, we deduce (with the notation $\Lambda^s = \Lambda^\Phi$) that

$$\Lambda^s(v,v_*) \le s\Big[s\,|v|^{2s-2}\,|v_*|^2 - |v_*|^{2s} + |v||v_*|\big(|v|^{2s-2} + |v_*|^{2s-2}\big)\Big]. \tag{2.3.9}$$

As $s>1$, we have $2s-1>1$ and Young's inequality ensures that

$$x^{2s-2}\,y^2 = x^{2s-2}\,y^{(2s-2)/(2s-1)}\,y^{2s/(2s-1)} \le \frac{2s-2}{2s-1}\,x^{2s-1}\,y + \frac{1}{2s-1}\,y^{2s}.$$

Substituting this inequality for $x=|v|$, $y=|v_*|$ into (2.3.9) yields

$$\Lambda^s(v,v_*) \le s\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1} - \frac{s-1}{2s-1}|v_*|^{2s}\Big].$$

Since $f\ge 0$ and $|v-v_*|^\gamma \le |v|^\gamma + |v_*|^\gamma$ $(\gamma\ge 0)$, we finally obtain

$$\begin{aligned}\frac{d}{dt}\int f(t,v)\,|v|^{2s}\,dv + 4\,s\,\frac{s-1}{2s-1}\iint |v-v_*|^\gamma\,|v_*|^{2s}\,f f_*(1-f_*)\,dv\,dv_* \\ \le 4s\iint f f_*\left(|v|^\gamma + |v_*|^\gamma\right)\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1}\Big]\,dv\,dv_*.\end{aligned}$$

Now,

$$\begin{aligned}\left(|v|^\gamma + |v_*|^\gamma\right)\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1}\Big] \;\le\; & (s+1)\Big[|v|^{2s+\gamma-1}\,|v_*| + |v|^{2s-1}\,|v_*|^{1+\gamma}\Big] \\ & + \Big[|v|^{1+\gamma}|v_*|^{2s-1} + |v||v_*|^{2s+\gamma-1}\Big],\end{aligned}$$

and Young's inequality ensures that

$$\max\Big\{|v|,|v|^{\gamma+1}\Big\} \le 1+|v|^2 \quad \text{and} \quad \max\Big\{|v|^{2s-1},|v|^{2s+\gamma-1}\Big\} \le 1+|v|^{2s}.$$

Therefore,

$$\frac{d}{dt}M_{2s}(f) + 4\,s\,\frac{s-1}{2s-1}\iint |v-v_*|^\gamma\,|v_*|^{2s}\,f f_*(1-f_*)\,dv\,dv_* \le C_1(s) + C_2(s)\,M_{2s}(f). \tag{2.3.10}$$

Thanks to the Gronwall lemma, we first conclude that, for every $T\ge 0$,

$$M_{2s}(f)(t) \le \left(M_{2s}(f_{in}) + 1\right)C_3(s,T), \qquad t\in[0,T]. \tag{2.3.11}$$

We next integrate (2.3.10) over $(0,T)$ and deduce from (2.3.11) that

$$\int_0^T \iint |v-v_*|^\gamma\,|v_*|^{2s}\,f f_*(1-f_*)\,dv\,dv_*\,d\tau \le \left(M_{2s}(f_{in}) + 1\right)C_4(s,T).$$

Since $|v-v_*|^\gamma \geq |v_*|^\gamma - |v|^\gamma$, we infer that

$$\|f_{in}\|_{L^1} \int_0^T \int f_*(1-f_*)|v_*|^{2s+\gamma}\, dv_* \leq (M_{2s}(f_{in})+1)\, C_4(s,T) + T\, \|f_{in}\|_{L^1_2}\, \|f\|_{L^\infty(0,T;L^1_{2s})}, \tag{2.3.12}$$

which completes the proof. □

Remark 2.3.3 *Unlike the classical Landau equation for which $M_{2s}(f)$ becomes instantaneously finite for positive times and $s>1$, we obtain here the propagation of these moments but not their appearance. This is due to the term $f_*(1-f_*)$ in (2.3.12). Consequently, we do not recover the same smoothness as in [10, Theorems 3 and 5].*

Lemma 2.3.4 *For every $T>0$, there exists a constant $C(s,T)$ such that*

$$K \int_0^T \int |\nabla f|^2 \left(1+|v|^2\right)^{s+\gamma/2} dv\, d\tau \\ \leq C(s,T) \left[1 + \|f(1-f)\|_{L^1(0,T;L^1_{2+\gamma})}\right] \|f\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \tag{2.3.13}$$

Proof. Let $s \geq 0$ and $f_{in} \in L^1_{2s}(\mathbb{R}^3)$. Since $0 \leq f \leq 1$, (2.2.4), (2.2.5) and Lemma 2.3.2 imply that $f \in L^\infty\big(0,T;L^2_{2s}(\mathbb{R}^3)\big)$. It follows from (2.2.1) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\big(1+|v|^2\big)^s dv &= -\int \Big(\overline{A}\nabla f - \overline{b} f(1-f)\Big)\nabla\Big(f\big(1+|v|^2\big)^s\Big)\, dv \\ &= -\int \overline{A}\nabla f \nabla f\big(1+|v|^2\big)^s dv - 2\,s \int \overline{A} f \nabla f\, v\big(1+|v|^2\big)^{s-1} dv \\ &\quad + \int f(1-f)\overline{b}\cdot\nabla f\big(1+|v|^2\big)^s dv + 2\,s\int \overline{b}\cdot v\, f^2(1-f)\big(1+|v|^2\big)^{s-1} dv. \end{aligned} \tag{2.3.14}$$

On the one hand, since $S(f)$ is a non-decreasing function and f satisfies (2.2.5), Proposition 2.2.3 implies that

$$\int \overline{A}\nabla f\nabla f\big(1+|v|^2\big)^s dv \geq K \int |\nabla f|^2 \left(1+|v|^2\right)^{s+\gamma/2} dv.$$

On the other hand, it is easy to see that there exists a constant C such that

$$\begin{aligned} \left|\nabla\cdot\left(\overline{A}v\left(1+|v|^2\right)^{s-1}\right)\right| &\leq C\,\|f(1-f)\|_{L^1_{2+\gamma}}\left(1+|v|^2\right)^{s+\gamma/2}, \\ \left|\nabla\cdot\big(\overline{b}\big(1+|v|^2\big)^s\big)\right| &\leq C\,\|f\|_{L^1_2}\left(1+|v|^2\right)^{s+\gamma/2}, \\ |\overline{b}| &\leq C\,\|f\|_{L^1_2}\left(1+|v|^2\right)^{(1+\gamma)/2}, \end{aligned} \tag{2.3.15}$$

so that

$$\int \overline{b}\cdot v\, f^2(1-f)\big(1+|v|^2\big)^{s-1} dv \leq C\,\|f\|_{L^1_2} \int f^2\big(1+|v|^2\big)^{s+\gamma/2} dv,$$

and

$$\begin{aligned}\int f(1-f)\,\overline{b}\cdot\nabla f\big(1+|v|^2\big)^s dv &= -\int\left(\frac{1}{2}f^2-\frac{1}{3}f^3\right)\nabla\cdot\Big(\overline{b}\big(1+|v|^2\big)^s\Big)dv\\ &\le C\,\|f\|_{L^1_2}\int f^2\big(1+|v|^2\big)^{s+\gamma/2}dv,\\ -2\int\overline{A}f\nabla f\,v\,\big(1+|v|^2\big)^{s-1}dv &= \int f^2\,\nabla\cdot\Big[\overline{A}\,v\,\big(1+|v|^2\big)^{s-1}\Big]\,dv\\ &\le C\,\|f(1-f)\|_{L^1_{2+\gamma}}\int f^2\big(1+|v|^2\big)^{s+\gamma/2}dv.\end{aligned}$$

Substituting the previous estimates into (2.3.14) and using (2.2.4) and (2.2.5) yield (2.3.13) after integrating with respect to time. □

2.4 Existence

This section is devoted to the proof of the existence part of Theorem 2.2.2. First we investigate a regularized problem and show the existence and smoothness of a solution. Indeed, a first difficulty common to both the Landau and LFD equations lies in the fact that the coefficients of the elliptic operator $Q_L(f)$ are unbounded. We thus approximate them by bounded ones. However, the coefficients remain non-local, which is the second difficulty to be faced. The existence of approximated solutions follows from a fixed point method but, unlike the classical Landau equation, this method has to be applied to a nonlinear equation. Finally, we obtain solutions to the LFD equation as cluster points of sequences of approximated solutions. At this stage, owing to the cubic nature of the LFD equation, weak convergence is not sufficient.

2.4.1 The regularized problem

Let $(\Psi_\varepsilon)_{\varepsilon>0}$ be a family of smooth bounded functions on $\mathbb{R}_+$ which coincide with $\Psi(r) = r^{\gamma+2}$ for $0<\varepsilon<r<\varepsilon^{-1}$ and enjoy the following properties.

- *(i)* The functions $\Psi'_\varepsilon, \Psi''_\varepsilon, \Psi^{(3)}_\varepsilon$ and $\Psi^{(4)}_\varepsilon$ are bounded;
- *(ii)* For $0<r<\varepsilon^{-1}$, $\quad \Psi_\varepsilon(r)\ge r^{\gamma+2}/2$;
  For $r>\varepsilon^{-1}$, $\quad \Psi_\varepsilon(r)\ge \varepsilon^{-(\gamma+2)}/2>0$;
- *(iii)* For every $r>0$, $\quad \Psi_\varepsilon(r)\le r^2(1+r^\gamma)$ and $|\Psi'_\varepsilon(r)|\le(\gamma+2)\,r\,(1+r^\gamma)$;
- *(iv)* For $0<r<\varepsilon$, $\Psi_\varepsilon(r)=r^2\,\nu_\varepsilon(r)$, with $\nu_\varepsilon\in\mathcal{C}^\infty([0,\varepsilon])$, $\nu_\varepsilon(0)=1$, $\nu'_\varepsilon(0)=0$ and $\nu''_\varepsilon(0)=0$.

For $(i,j) \in [\![1,3]\!]^2$, we set

$$a^\varepsilon(z) = \big(a^\varepsilon_{i,j}(z)\big)_{i,j} \qquad \text{with} \qquad a^\varepsilon_{i,j}(z) = \Psi_\varepsilon(|z|)\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right),$$

$$b^\varepsilon_i(z) = \sum_k \partial_k a^\varepsilon_{i,k}(z) = -\,\frac{2\,z_i}{|z|^2}\,\Psi_\varepsilon(|z|),$$

$$c^\varepsilon(z) = \sum_{k,l} \partial^2_{kl} a^\varepsilon_{k,l}(z) = -\,\frac{2}{|z|^2}\Big[\Psi_\varepsilon(|z|) + |z|\,\Psi'_\varepsilon(|z|)\Big],$$

and consider the regularized problem

$$\partial_t f \;=\; \nabla\cdot\Big(\overline{A}^{f,\varepsilon}\nabla f - \overline{b}^{f,\varepsilon} f(1-f)\Big) + \varepsilon\Delta f, \tag{2.4.1}$$

$$f(0,.) \;=\; f_{in}. \tag{2.4.2}$$

We first note that, thanks to the properties of Ψ_ε, we have the following result.

Lemma 2.4.1 *The functions $a^\varepsilon_{i,j}$ and b^ε_i belong to $C^4_b(\mathbb{R}^3)$. The function c^ε belongs to $C^2_b(\mathbb{R}^3)$.*

We set

$$K_\varepsilon = \max_{i,j} \|a^\varepsilon_{i,j}\|_{C^4_b} + \max_i \|b^\varepsilon_i\|_{C^4_b} + \|c^\varepsilon\|_{C^2_b}.$$

We next investigate the well-posedness of (2.4.1), (2.4.2).

Theorem 2.4.2 *Consider $f_{in} \in \mathcal{C}^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap W^{3,\infty}(\mathbb{R}^3)$ such that*

$$0 < \alpha_1\, e^{-\beta_1|v|^2} \le f_{in}(v) \le \frac{\alpha_2\, e^{-\beta_2|v|^2}}{1+\alpha_2\, e^{-\beta_2|v|^2}} < 1 \qquad \text{for every } v\in\mathbb{R}^3, \tag{2.4.3}$$

for positive constants α_1, α_2, β_1 and β_2. Let $\varepsilon > 0$ and $T > 0$. Then, there exists a solution f^ε to the regularized problem (2.4.1), (2.4.2) with initial condition f_{in} such that, for every $s > 0$, f^ε belongs to $L^\infty\big(0,T;L^1_{2s}(\mathbb{R}^3)\big) \cap L^2\big(0,T;H^1_{2s}(\mathbb{R}^3)\big)$.

Let $\beta_1' \ge \beta_1$, D, E, F and C_L be five positive constants, the values of which we will specify later. We denote by $\mathcal{C}$ the set of functions $f \in \mathcal{C}([0,T];L^1(\mathbb{R}^3))$ such that, for all $s,t \in [0,T]$ and $\varphi \in \mathcal{C}^2_b(\mathbb{R}^3)$,

$$0 \le f \;\le\; 1,$$

$$\int f(t,v)\,dv \;=\; \int f_{in}(v)\,dv,$$

$$\left|\int \big(f(t,v) - f(s,v)\big)\,\varphi(v)\,dv\right| \;\le\; C_L\,\|\varphi\|_{\mathcal{C}^2_b}\,|t-s|,$$

$$\left|\int \big(f(1-f)(t,v) - f(1-f)(s,v)\big)\,\varphi(v)\,dv\right| \;\le\; C_L\,\|\varphi\|_{\mathcal{C}^2_b}\,|t-s|,$$

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f(t,v) \;\le\; \frac{\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}.$$

For $g \in \mathcal{C}$, we consider the following quasi-linear problem

$$\partial_t f = \nabla \cdot \Big[\big(\overline{A}^{g,\varepsilon} + \varepsilon I_3 \big) \nabla f - \overline{b}^{g,\varepsilon} f(1-f) \Big], \tag{2.4.4}$$
$$f(0,.) = f_{in}, \tag{2.4.5}$$

where I_3 denotes the identity matrix of $\mathbb{R}^3$.

The existence of solutions to (2.4.1), (2.4.2) will follow from the existence of solutions to (2.4.4), (2.4.5) by means of a fixed point method. We thus first study the latter and prove the following result.

Theorem 2.4.3 *Let $\delta \in (0,1)$ and $\varepsilon > 0$. For each $g \in \mathcal{C}$, there exists a unique classical solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}\big([0,T] \times \mathbb{R}^3\big)$ to (2.4.4), (2.4.5) and there is a constant Λ depending only on f_{in}, δ, T, ε and C_L such that*

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,\,(2+\delta)/2}} \leq \Lambda.$$

Moreover, there exist constants β_1', D, E, F and C_L depending only on f_{in}, T and ε such that f^ε belongs to $\mathcal{C}$.

For $T > 0$, $l > 0$, $l \notin \mathbb{N}$ and Ω a domain of $\mathbb{R}^3$, we consider Hölder spaces $\mathcal{H}^{l,\,l/2}([0,T] \times \Omega)$, whose norm are

$$\begin{aligned} \|f\|_{\mathcal{H}^{l,l/2}} = & \sup_{0 \leq t \leq T,\, v \in \mathbb{R}^3} \sum_{|\alpha|+2r \leq [l]} |\partial_t^r \partial_\alpha f(t,v)| \\ & + \sup_{0 \leq t \leq T,\, v \neq w} \sum_{|\alpha|+2r=[l]} \frac{|\partial_t^r \partial_\alpha f(t,v) - \partial_t^r \partial_\alpha f(t,w)|}{|v-w|^{l-[l]}} \\ & + \sup_{s \neq t,\, v \in \mathbb{R}^3} \sum_{|\alpha|+2r=[l]} \frac{|\partial_t^r \partial_\alpha f(t,v) - \partial_t^r \partial_\alpha f(s,v)|}{|t-s|^{(l-[l])/2}}, \end{aligned}$$

where $[l]$ denotes the integer part of l and $\alpha \in \mathbb{N}^3$.

Thanks to Lemma 2.4.1 and to the properties of $\mathcal{C}$, the coefficients of the parabolic operator in (2.4.4) have the following regularity properties.

Lemma 2.4.4 *Let $\varepsilon > 0$, $\delta \in (0,1)$ and $g \in \mathcal{C}$. For every $(i,j,k) \in [\![1,3]\!]^3$, the functions $\overline{A}_{i,j}^{g,\varepsilon}$, $\overline{b}_i^{g,\varepsilon}$, $\partial_k \overline{A}_{i,j}^{g,\varepsilon}$ and $\overline{c}^{g,\varepsilon}$ belong to the Hölder space $\mathcal{H}^{\delta,\,\delta/2}\big([0,T] \times \mathbb{R}^3\big)$, with*

$$\max_{i,j} \|\overline{A}_{i,j}^{g,\varepsilon}\|_{W^{1,\infty}} + \max_i \|\overline{b}_i^{g,\varepsilon}\|_{L^\infty} + \|\overline{c}^{g,\varepsilon}\|_{L^\infty} \leq K_\varepsilon \|f_{in}\|_{L^1}. \tag{2.4.6}$$

Moreover, for every bounded domain Ω of $\mathbb{R}^3$, the functions $\overline{A}_{i,j}^{g,\varepsilon}$, $\overline{b}_i^{g,\varepsilon}$, $\partial_k \overline{A}_{i,j}^{g,\varepsilon}$ and $\partial_k \overline{b}_i^{g,\varepsilon}$ belong to the Hölder space $\mathcal{H}^{1+\delta,\,(1+\delta)/2}\big([0,T] \times \Omega\big)$.

Proof of Theorem 2.4.3. Owing to the uniform ellipticity

$$\varepsilon\,|\xi|^2 \le \sum_{i,j} (\overline{A}^{g,\varepsilon}_{i,j}(v) + \varepsilon\delta_{i,j})\,\xi_i\xi_j \le (\,3\,K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon\,)\,|\xi|^2, \qquad v\in\mathbb{R}^3, \quad \xi\in\mathbb{R}^3 \quad (2.4.7)$$

and classical arguments, the maximum principle and [17, Theorem 5.8.1] imply the existence and uniqueness of a solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ to (2.4.4), (2.4.5). This solution satisfies

$$0 \le f^\varepsilon(t,v) \le 1, \tag{2.4.8}$$

and there exists a constant Λ depending only on f_{in}, δ, T, ε and C_L such that

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,(2+\delta)/2}} \le \Lambda,$$

(see the Appendix for a sketch of proof).

We next show that we can choose constants β_1', D, E, F and C_L such that $f^\varepsilon \in \mathcal{C}$. First, we verify that, for every $(t,v)\in[0,T]\times\mathbb{R}^3$,

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f^\varepsilon(t,v) \le \frac{\alpha_2\,e^{Et}\,e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\,e^{Et}\,e^{-\beta_2\,|v|^2/(1+Ft)}}. \tag{2.4.9}$$

Indeed, introducing

$$\varphi_{inf}(t,v) = \alpha_1\,e^{-\beta_1'|v|^2}\,e^{-Dt},$$

and the parabolic operator $\mathcal{L}$ defined by

$$\mathcal{L}u = \partial_t u - \sum_{i,j}\left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\,\delta_{i,j}\right)\partial^2_{i,j}u - \sum_i \left[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2f^\varepsilon)\right]\partial_i u + \overline{c}^{g,\varepsilon}\,(1-f^\varepsilon)\,u,$$

we see that $\mathcal{L}\varphi_{inf} \le 0$ as soon as

$$\beta_1' \ge \frac{1}{4\varepsilon} \quad \text{and} \quad D \ge 12\beta_1' K_\varepsilon^2 \|f_{in}\|^2_{L^1} + 6\beta_1'(K_\varepsilon\|f_{in}\|_{L^1}+\varepsilon) + K_\varepsilon\|f_{in}\|_{L^1},$$

whence

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f^\varepsilon(t,v), \qquad \forall (t,v)\in[0,T]\times\mathbb{R}^3,$$

by the comparison principle [17, Theorem 1.2.1].

We next set

$$\varphi_{sup}(t,v) = \frac{\alpha_2\,e^{Et}\,e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\,e^{Et}\,e^{-\beta_2\,|v|^2/(1+Ft)}},$$

and let $\mathcal{M}$ be the semilinear operator defined by

$$\mathcal{M}u = \partial_t u - \sum_{i,j}\left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\,\delta_{i,j}\right)\partial^2_{i,j}u - \sum_i \left[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2u)\right]\partial_i u + \overline{c}^{g,\varepsilon}(1-u)\,u.$$

For

$$E \ge 12K_\varepsilon^2\,\|f_{in}\|_{L^1} + K_\varepsilon\,\|f_{in}\|^2_{L^1} \quad \text{and} \quad F \ge 12\,\beta_2 K_\varepsilon \|f_{in}\|_{L^1} + 4\,\beta_2\,\varepsilon + \beta_2,$$

we have $\mathcal{M}\varphi_{sup} \geq 0 = \mathcal{M}f^\varepsilon$. Owing to the regularity of the coefficients of the parabolic operator, we are in a position to apply the comparison principle [19, Theorem 9.1] to obtain that

$$f^\varepsilon(t,v) \leq \frac{\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}.$$

It readily follows from (2.4.9) and the continuity of f^ε that $f^\varepsilon \in \mathcal{C}([0,T];L^1(\mathbb{R}^3))$. In addition, classical truncation arguments, (2.4.6) and (2.4.9) allow us to check that

$$\int f^\varepsilon(t,v)\,dv = \int f_{in}(v)\,dv, \qquad t \in [0,T]. \tag{2.4.10}$$

It remains now to verify the two Lipschitz properties and this will be the aim of the three following lemmas. We only give formal calculations but they can be rigorously justified by standard truncation arguments.

Lemma 2.4.5 *For every $r \geq 0$, f^ε belongs to $L^2(0,T;H^1_{2r}(\mathbb{R}^3))$. Moreover, there exists a constant C depending only on f_{in}, r, ε and T such that,*

$$\|f^\varepsilon\|_{L^2(0,T;H^1_{2r})} \leq C.$$

Proof. Let $r \geq 0$. We multiply (2.4.1) by $f^\varepsilon\big(1+|v|^2\big)^r$ and we integrate with respect to v to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |f^\varepsilon|^2(t,v)\big(1+|v|^2\big)^r\,dv \;=\;& -\int (\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla f^\varepsilon \nabla f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\\
& -2r\int (\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla f^\varepsilon\, v\, f^\varepsilon\,\big(1+|v|^2\big)^{r-1}\,dv\\
& +\int f^\varepsilon(1-f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot\nabla f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\\
& +2r\int (f^\varepsilon)^2\,(1-f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot v\,\big(1+|v|^2\big)^{r-1}\,dv.
\end{aligned}$$

After integrating over $(0,t)$, we infer from (2.4.6), (2.4.7) and Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\int |f^\varepsilon|^2(t,v)\big(1+|v|^2\big)^r\,dv + \varepsilon\int_0^t\int |\nabla f^\varepsilon|^2\big(1+|v|^2\big)^r\,dv\,d\tau\\
&\leq \frac{\varepsilon}{3}\int_0^t\int |\nabla f^\varepsilon|^2\big(1+|v|^2\big)^r\,dv\,d\tau + C_\varepsilon[3\,K_\varepsilon\|f_{in}\|_{L^1}+\varepsilon]^2\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^{r-1}\,dv\,d\tau\\
&\quad+\frac{\varepsilon}{3}\int_0^t\int |\nabla f^\varepsilon|^2\big(1+|v|^2\big)^r\,dv\,d\tau + C_\varepsilon K_\varepsilon^2\|f_{in}\|_{L^1}^2\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\,d\tau\\
&\quad+CK_\varepsilon\|f_{in}\|_{L^1}\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\,d\tau + \frac{1}{2}\int |f_{in}|^2\big(1+|v|^2\big)^r\,dv.
\end{aligned}$$

Therefore,

$$\varepsilon \int_0^t \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^r dv\, d\tau \\ \leq C(\varepsilon, M_{in}) \int_0^t \int f^\varepsilon \left(1+|v|^2\right)^r dv\, d\tau + \frac{3}{2} \int |f_{in}|^2 \left(1+|v|^2\right)^r dv,$$

and (2.4.9) implies that the right-hand side of the above inequality is bounded. □

Lemma 2.4.6 *The function f^ε belongs to $L^\infty\big(0,T;H^1(\mathbb{R}^3)\big)$. Moreover, there exists a constant G depending only on f_{in}, ε and T such that,*

$$\|f^\varepsilon\|_{L^\infty(0,T;H^1)} \leq G.$$

Proof. We first observe that

$$f^\varepsilon \in \mathcal{H}^{2+\delta,(2+\delta)/2}([0,T]\times\mathbb{R}^3) \cap \mathcal{H}^{3+\delta,\,(3+\delta)/2}([0,T]\times\Omega)$$

for each bounded domain $\Omega \subset \mathbb{R}^3$ by [17, Theorem 5.8.1]. We may thus differentiate (2.4.4) with respect to v_k and obtain

$$\partial_t\partial_k f^\varepsilon = \nabla\cdot\Big[(\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla\partial_k f^\varepsilon + \partial_k\overline{A}^{g,\varepsilon}\nabla f^\varepsilon - \overline{b}^{g,\varepsilon}(1-2f^\varepsilon)\partial_k f^\varepsilon - \partial_k\overline{b}^{g,\varepsilon} f^\varepsilon(1-f^\varepsilon)\Big].$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(\partial_k f^\varepsilon)^2\,dv \;=\; & -\int(\overline{A}^{g,\varepsilon}+\varepsilon I_3)\,\nabla\partial_k f^\varepsilon\,\nabla\partial_k f^\varepsilon\,dv - \int \partial_k\overline{A}^{g,\varepsilon}\,\nabla f^\varepsilon\,\nabla\partial_k f^\varepsilon\,dv \\ & + \int(1-2f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot\nabla\partial_k f^\varepsilon\,\partial_k f^\varepsilon\,dv + \int f^\varepsilon(1-f^\varepsilon)\,\partial_k\overline{b}^{g,\varepsilon}\cdot\nabla\partial_k f^\varepsilon\,dv,\end{aligned}$$

and (2.4.6), (2.4.7) and Young's inequality lead to

$$\int |\partial_k f^\varepsilon|^2\,dv + 2\varepsilon\int_0^t\int|\nabla\partial_k f^\varepsilon|^2\,dv\,d\tau \leq \frac{3\varepsilon}{2}\int_0^t\int|\nabla\partial_k f^\varepsilon|^2\,dv\,d\tau \\ + C_\varepsilon K_\varepsilon^2\,\|f_{in}\|_{L^1}^2\,\|\nabla f^\varepsilon\|^2_{L^2(0,T;\,L^2)} + C_\varepsilon T\,K_\varepsilon^2\,\|f_{in}\|^3_{L^1} + \|f_{in}\|^2_{H^1}.$$

Lemma 2.4.6 then readily follows from the above inequality by Lemma 2.4.5. □

Lemma 2.4.7 *There exists a constant C_L depending only on f_{in}, ε and T such that, for all $\varphi\in\mathcal{C}_b^2(\mathbb{R}^3)$ and $\sigma,t\in[0,T]$,*

$$\begin{aligned}\left|\int\big(f^\varepsilon(t,v)-f^\varepsilon(\sigma,v)\big)\,\varphi(v)\,dv\right| \;&\leq\; C_L\,\|\varphi\|_{\mathcal{C}_b^2}\,|t-\sigma|,\\ \left|\int\big(f^\varepsilon(1-f^\varepsilon)(t,v)-f^\varepsilon(1-f^\varepsilon)(\sigma,v)\big)\,\varphi(v)\,dv\right| \;&\leq\; C_L\,\|\varphi\|_{\mathcal{C}_b^2}\,|t-\sigma|.\end{aligned}$$

Proof. Let $\varphi \in \mathcal{C}_b^2(\mathbb{R}^3)$. Classical truncation arguments ensure that

$$\int f^\varepsilon(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(\sigma,v)\,\varphi(v)\,dv$$
$$= \int_\sigma^t d\tau \Big\{ \sum_{i,j} \int \overline{A}^{g,\varepsilon}_{i,j}\, f^\varepsilon\, \partial^2_{i,j}\varphi\, dv + \int f^\varepsilon\, \overline{B}^{g,\varepsilon} \cdot \nabla\varphi\, dv$$
$$+ \int f^\varepsilon \big(1-f^\varepsilon\big)\, \overline{b}^{g,\varepsilon} \cdot \nabla\varphi\, dv + \varepsilon \int f^\varepsilon\, \Delta\varphi\, dv \Big\}, \quad (2.4.11)$$

The first inequality of Lemma 2.4.7 then readily follows from (2.4.10), (2.4.11) and Lemma 2.4.4 with

$$C_L \geq C_1 = C\,(K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon)\, \|f_{in}\|_{L^1}.$$

Similarly, we infer from (2.4.4) that, for $\varphi \in \mathcal{C}_b^2(\mathbb{R}^3)$, we have

$$\int f^\varepsilon(1-f^\varepsilon)(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(1-f^\varepsilon)(\sigma,v)\,\varphi(v)\,dv$$
$$= \int_\sigma^t d\tau \Big\{ - \int (\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla f^\varepsilon \Big[\big(1-2f^\varepsilon\big)\nabla\varphi - 2\,\varphi \nabla f^\varepsilon \Big]\, dv$$
$$-2 \int f^\varepsilon \big(1-f^\varepsilon\big)\, \varphi\, \overline{b}^{g,\varepsilon} \cdot \nabla f^\varepsilon\, dv + \int f^\varepsilon \big(1-f^\varepsilon\big)\big(1-2\,f^\varepsilon\big)\, \overline{b}^{g,\varepsilon} \cdot \nabla\varphi\, dv \Big\}. \quad (2.4.12)$$

With the notation $M_1 : M_2 = \sum_{i,j} M_{1i,j} M_{2i,j}$ for any two matrices M_1 and M_2, we have

$$\int (\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla f^\varepsilon\, \big(1-2f^\varepsilon\big)\nabla\varphi\, dv = - \int f^\varepsilon \big(1-f^\varepsilon\big)\, \nabla \cdot \Big((\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla\varphi \Big)\, dv$$
$$= - \int f^\varepsilon \big(1-f^\varepsilon\big)\, \overline{B}^{g,\varepsilon} \cdot \nabla\varphi\, dv - \int f^\varepsilon \big(1-f^\varepsilon\big)\, (\overline{A}^{g,\varepsilon} + \varepsilon I_3) : \nabla^2\varphi\, dv,$$

and the identity (2.4.12) becomes

$$\int f^\varepsilon(1-f^\varepsilon)(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(1-f^\varepsilon)(\sigma,v)\,\varphi(v)\,dv = \int_\sigma^t d\tau \Big\{ \int f^\varepsilon \big(1-f^\varepsilon\big)\, \overline{B}^{g,\varepsilon} \cdot \nabla\varphi\, dv$$
$$+ \int f^\varepsilon \big(1-f^\varepsilon\big)\, (\overline{A}^{g,\varepsilon} + \varepsilon I_3) : \nabla^2\varphi\, dv + 2 \int (\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla f^\varepsilon\, \nabla f^\varepsilon\, \varphi\, dv$$
$$-2 \int f^\varepsilon \big(1-f^\varepsilon\big)\, \varphi\, \overline{b}^{g,\varepsilon} \cdot \nabla f^\varepsilon\, dv + \int f^\varepsilon \big(1-f^\varepsilon\big)\, \big(1-2\,f^\varepsilon\big)\, \overline{b}^{g,\varepsilon} \cdot \nabla\varphi\, dv \Big\}.$$

The second inequality of Lemma 2.4.7 then follows with the help of (2.4.10) and Lemmas 2.4.4 and 2.4.6 with

$$C_L \geq C_2 = C\,(K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon)\, (\|f_{in}\|_{L^1} + G^2).$$

Choosing $C_L = \max(C_1, C_2)$ completes the proof of Lemma 2.4.7. □

We have thus found β_1', D, E, F and C_L depending only on f_{in}, T and ε such that, if $g \in \mathcal{C}$, $f^\varepsilon \in \mathcal{C}$ and the proof of Theorem 2.4.3 is complete. □

Proof of Theorem 2.4.2. We fix β_1', D, E, F and C_L as in Theorem 2.4.3. For $g \in \mathcal{C}$, we denote by $\Phi(g)$ the unique solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ to (2.4.4), (2.4.5). Then $\Phi(g) \in \mathcal{C}$ by Theorem 2.4.3 and we now check that $\Phi : \mathcal{C} \to \mathcal{C}$ is continuous and compact for the topology of $\mathcal{C}\big([0,T];L^1(\mathbb{R}^3)\big)$.

Continuity of Φ. Consider $g_1 \in \mathcal{C}$, $g_2 \in \mathcal{C}$ and put $f_i^\varepsilon = \Phi(g_i)$ for $i = 1,2$. Then, $u = f_1^\varepsilon - f_2^\varepsilon$ satisfies

$$\partial_t u - \sum_{i,j}\big(\overline{A}_{i,j}^{g_1,\varepsilon} + \varepsilon\delta_{i,j}\big)\partial^2_{i,j}u - \sum_j\Big[\overline{B}_j^{g_1,\varepsilon} - \overline{b}_j^{g_1,\varepsilon}(1-f_1^\varepsilon - f_2^\varepsilon)\Big]\partial_j u$$
$$+\Big[\overline{c}^{g_1,\varepsilon}(1-f_1^\varepsilon-f_2^\varepsilon) - \sum_j \overline{b}_j^{g_1,\varepsilon}(\partial_j f_1^\varepsilon + \partial_j f_2^\varepsilon)\Big]u = \Upsilon,$$

where

$$\Upsilon = \sum_{i,j}\Big(\overline{A}_{i,j}^{g_1,\varepsilon} - \overline{A}_{i,j}^{g_2,\varepsilon}\Big)\partial^2_{i,j}f_2^\varepsilon + \sum_j\Big(\overline{B}_j^{g_1,\varepsilon} - \overline{B}_j^{g_2,\varepsilon}\Big)\partial_j f_2^\varepsilon$$
$$-\sum_j\Big(\overline{b}_j^{g_1,\varepsilon} - \overline{b}_j^{g_2,\varepsilon}\Big)(1-2f_2^\varepsilon)\partial_j f_2^\varepsilon - \Big(\overline{c}^{g_1,\varepsilon} - \overline{c}^{g_2,\varepsilon}\Big)f_2^\varepsilon(1-f_2^\varepsilon).$$

Since u belongs to $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ and is bounded ($|u|\le 1$), we infer from the maximum principle [17, Theorem 1.2.5] that

$$\sup_{[0,T]\times\mathbb{R}^3}\big|f_1^\varepsilon - f_2^\varepsilon\big| \le \left(\sup_{\mathbb{R}^3}\big|f_1^\varepsilon - f_2^\varepsilon\big|(0) + T\max_{[0,T]\times\mathbb{R}^3}|\Upsilon|\right)e^{\omega T},$$

with

$$\omega = K_\varepsilon\,\|f_{in}\|_{L^1}(1+6\Lambda),$$

where the constant Λ is given by Theorem 2.4.3. Since $f_1^\varepsilon(0,.) = f_{in} = f_2^\varepsilon(0,.)$ and

$$|\Upsilon| \le K_\varepsilon\big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)}\Big(\sum_{i,j}\sup|\partial^2_{i,j}f_2^\varepsilon| + 4\sum_j\sup|\partial_j f_2^\varepsilon| + 1\Big),$$

we deduce that

$$\sup_{[0,T]\times\mathbb{R}^3}\big|f_1^\varepsilon - f_2^\varepsilon\big| \le CTe^{\omega T}\big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)}.$$

Now, for $R>0$, we have

$$\begin{aligned}\big|f_1^\varepsilon - f_2^\varepsilon\big|_{\mathcal{C}([0,T];L^1)} &\le \sup_{t\in[0,T]}\int_{|v|\le R}|f_1^\varepsilon - f_2^\varepsilon|(t,v)\,dv + \sup_{t\in[0,T]}\int_{|v|\ge R}\big(|f_1^\varepsilon| + |f_2^\varepsilon|\big)(t,v)\,dv\\ &\le C\,R^3\,T\,e^{\omega T}\big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)} + 2\,\alpha_2\,e^{ET}\int_{|v|\ge R}e^{-\beta_2|v|^2/(1+FT)}\,dv\\ &\le C(T)R^3\big|g_1-g_2\big|_{\mathcal{C}([0,T];L^1)} + \frac{C(T)}{R^3},\end{aligned}$$

whence

$$\left|f_1^\varepsilon - f_2^\varepsilon\right|_{\mathcal{C}([0,T];L^1)} \leq C(T)\left|g_1 - g_2\right|_{\mathcal{C}([0,T];L^1)}^{1/2},$$

with the choice

$$R = \left|g_1 - g_2\right|_{\mathcal{C}([0,T];L^1)}^{-1/6}.$$

Compactness of Φ. For $m \geq 4$, we have $L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3) \subset (H^m(\mathbb{R}^3))'$ with a compact embedding $L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3)$. Since,

$$\begin{array}{c} \Phi(\mathcal{C}) \text{ is bounded in } L^\infty(0,T; L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)) \\ \text{and } \partial_t \Phi(\mathcal{C}) \text{ is bounded in } L^r(0,T; (H^m(\mathbb{R}^3))'), \quad \text{ with } r > 1, \end{array}$$

by Theorem 2.4.3, we deduce from [27, Corollary 4], that $\Phi(\mathcal{C})$ is relatively compact in $\mathcal{C}([0,T]; L^1(\mathbb{R}^3))$.

We are now in a position to complete the proof of Theorem 2.4.2. Indeed, $\mathcal{C}$ is a non-empty, convex, closed and bounded subset from the Banach space $\mathcal{C}\big([0,T]; L^1\big(\mathbb{R}^3\big)\big)$. Since Φ is a compact and continuous map from $\mathcal{C}$ into $\mathcal{C}$, the Schauder fixed point theorem ensures the existence of a fixed point of Φ, that is, of a solution to (2.4.1), (2.4.2). In addition, (2.4.9) and Lemma 2.4.5 warrant that f^ε has the desired properties. □

2.4.2 Uniform estimates

In order to pass to the limit as $\varepsilon \to 0$ in (2.4.4), (2.4.5) and obtain a solution to (2.2.1), (2.2.2), we first need to establish uniform estimates on f^ε which do not depend on ε. These estimates are actually similar to those listed in Section 2.3. In the following, we denote by C any constant depending only on γ, M_{in}, E_{in} and S_{in}.

Lemma 2.4.8 *For all $\sigma, t \in [0,T]$, $\sigma \leq t$, the function f^ε satisfies*

$$\int f^\varepsilon(t,v)dv = M_{in}, \tag{2.4.13}$$

$$\int f^\varepsilon(t,v)|v|^2 dv = E_{in} + 6\varepsilon M_{in} t \leq E_{in} + 6\varepsilon M_{in} T, \tag{2.4.14}$$

and

$$S_{in} \leq S(f^\varepsilon)(\sigma) \leq S(f^\varepsilon)(t), \tag{2.4.15}$$

where $M_{in} = M_0(f_{in})$, $E_{in} = M_2(f_{in})$ and $S_{in} = S(f_{in})$.

Proof. Since $f^\varepsilon \in \mathcal{C}$, the first equality holds true. It next follows from (2.4.6), (2.4.9) and (2.4.11) with $\varphi(v) = |v|^2$ that

$$\int f^\varepsilon(t,v)\,|v|^2\,dv - \int f_{in}(v)\,|v|^2\,dv = 6\,\varepsilon \int_0^t \int f^\varepsilon(\tau,v)\,dv\,d\tau,$$

whence (2.4.14) by (2.4.13).

Finally, since f^ε is differentiable with respect to time and satisfies $0 < f^\varepsilon(t,v) < 1$ for every $(t,v) \in [0,T] \times \mathbb{R}^3$, by (2.4.9), we have

$$\partial_t \Big[f^\varepsilon \ln f^\varepsilon + (1-f^\varepsilon)\ln(1-f^\varepsilon) \Big] = \Big[\ln f^\varepsilon - \ln(1-f^\varepsilon) \Big] \partial_t f^\varepsilon.$$

Therefore, thanks to (2.4.1),

$$\begin{aligned} S(f^\varepsilon)(t) &= S(f_{in}) + \int_0^t d\tau \int \Big[\overline{A}^\varepsilon \nabla f^\varepsilon - \overline{b}^\varepsilon f^\varepsilon (1-f^\varepsilon) \Big] \frac{\nabla f^\varepsilon}{f^\varepsilon(1-f^\varepsilon)}\, dv \\ &\quad + \varepsilon \int_0^t d\tau \int \frac{|\nabla f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)}\, dv \\ &= S(f_{in}) + \frac{1}{2} \int_0^t d\tau \iint a^\varepsilon(v-v_*) \Big(f_*^\varepsilon (1-f_*^\varepsilon) \nabla f^\varepsilon - f^\varepsilon(1-f^\varepsilon) \nabla f_*^\varepsilon \Big) \\ &\quad \left(\frac{\nabla f^\varepsilon}{f^\varepsilon(1-f^\varepsilon)} - \frac{\nabla f_*^\varepsilon}{f_*^\varepsilon(1-f_*^\varepsilon)} \right) dv_*\, dv + \varepsilon \int_0^t d\tau \int \frac{|\nabla f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)}\, dv. \end{aligned}$$

Since the matrix a^ε is non-negative, we conclude that the function $S(f^\varepsilon)$ is non-decreasing and (2.4.15) follows. □

We next consider the ellipticity of the diffusion matrix, the propagation of moments and the smoothness of f^ε. Proceeding as in the proof of Proposition 2.2.3 with the help of the properties of Ψ_ε, we first have the following results.

Proposition 2.4.9 *Denote by R_* the constant given by Lemma 2.3.1. For every $0 < \varepsilon \le (3R_*)^{-1}$, we have*

(i) Let $f \in \mathcal{Y}(E_{in}, S_{in})$. Then there exists a constant $K > 0$ depending only on γ, E_{in} and S_{in}, such that, for every $v \in \mathbb{R}^3$,

$$\sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\, \delta_{i,j} \right) \xi_i\, \xi_j \ge K \left(1+|v|^2\right)^{\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2 \right) \right]^{\gamma+2} |\xi|^2, \quad \xi \in \mathbb{R}^3.$$

(ii) If $f(1-f) \in L^1_{\gamma+2}(\mathbb{R}^3)$, then there exists a constant $C > 0$ depending only on $M_{\gamma+2}(f)$ and $M_0(f)$ such that, for every $v \in \mathbb{R}^3$,

$$0 \le \sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\, \delta_{i,j} \right) \xi_i\, \xi_j \le \left(C\left(1+|v|^{\gamma+2}\right) + \varepsilon \right) |\xi|^2, \quad \xi \in \mathbb{R}^3.$$

In fact, the proof of the first point also gives a uniform (with respect to ε) ellipticity estimate.

Corollary 2.4.10 *For $0 < \varepsilon \le (3R_*)^{-1}$, there exists a constant κ depending only on γ, E_{in} and S_{in}, such that, for every $f \in \mathcal{Y}(E_{in}, S_{in})$,*

$$\sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\, \delta_{i,j} \right) \xi_i\, \xi_j \ge \kappa\, \frac{|\xi|^2}{1+|v|^2}, \quad \xi \in \mathbb{R}^3, \quad v \in \mathbb{R}^3.$$

We next proceed as in the proof of Lemma 2.3.2 to show the following result.

Lemma 2.4.11 *For all $T > 0$, $s > 1$, there exists a constant Γ depending only on s, T and $\|f_{in}\|_{L^1_{2s}}$ such that*

$$\sup_{t\in[0,T]} \|f^\varepsilon(t)\|_{L^1_{2s}} + \int_0^T \iint \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^{2s}\, f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, dv\, dv_*\, d\tau \le \Gamma(\|f_{in}\|_{L^1_{2s}}). \tag{2.4.16}$$

Remark 2.4.12 *The constant Γ increases with $\|f_{in}\|_{L^1_{2s}}$.*

Finally, a proof similar to that of Lemma 2.3.4 leads to the following H^1-estimate.

Lemma 2.4.13 *For all $T > 0$, $\varepsilon \in (0,1)$, $s \ge 0$, there exists a constant $C > 0$ depending only on s and T such that*

$$\begin{aligned} K \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s+\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2\right)\right]^{\gamma+2} dv\, d\tau \\ \le C \int_0^T \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_*\, d\tau \\ + C\,(1+\varepsilon)\, \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \end{aligned} \tag{2.4.17}$$

In particular, for $s \in [0,1]$, we have, for every $\delta > 0$,

$$\begin{aligned} K \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s+\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2\right)\right]^{\gamma+2} dv\, d\tau \\ \le C\, \Gamma(\|f_{in}\|_{L^1_{2+\delta}}) + C\,(1+\varepsilon)\, \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \end{aligned} \tag{2.4.18}$$

Using Corollary 2.4.10 instead of Proposition 2.4.9 in the proof of Lemma 2.4.13 yields

Corollary 2.4.14 *For all $T > 0$, $\varepsilon \in (0,1)$, $s \ge 0$, there exists a constant $C > 0$ depending only on s and T such that*

$$\begin{aligned} \kappa \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s-1} dv\, d\tau \\ \le C \int_0^T \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_*\, d\tau \\ + C\,(1+\varepsilon) \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \end{aligned}$$

Proof of Lemma 2.4.13. A slight change to the proof of Lemma 2.3.4 is required here since we do not have an estimate on $f^\varepsilon(1-f^\varepsilon)$ in $L^1(0,T;L^1_{2+\gamma}(\mathbb{R}^3))$ because Ψ_ε is bounded. Thus, (2.3.15) has to be replaced by

$$\begin{aligned} \left| \int (f^\varepsilon)^2\, \nabla \cdot \left(\overline{A}^\varepsilon v \left(1+|v|^2\right)^{s-1} \right) dv \right| \\ \le C \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_* + C\, \|f^\varepsilon\|_{L^1_2}\, \|f^\varepsilon\|_{L^1_{2s+\gamma}}, \end{aligned}$$

which gives (2.4.17).

Let $s \leq 1$ and $\delta > 0$. We deduce from Lemma 2.4.11 with $s = 1 + \delta/2$, Young's inequality, (2.4.13) and (2.4.14) that

$$\begin{aligned}\int_0^t \iint f^\varepsilon f_*^\varepsilon (1 - f_*^\varepsilon)\, \frac{\Psi_\varepsilon(|v - v_*|)}{|v - v_*|^2}\, |v_*|^2 \left(1 + |v|^2\right)^{s-1} dv\, dv_*\, d\tau \\ \leq \int_0^t \iint f^\varepsilon f_*^\varepsilon (1 - f_*^\varepsilon)\, \frac{\Psi_\varepsilon(|v - v_*|)}{|v - v_*|^2} \left(1 + |v_*|^{2+\delta}\right) dv\, dv_*\, d\tau \\ \leq C\, T\, \|f^\varepsilon\|^2_{L^\infty(0,T;L^1_2)} + \Gamma(\|f_{in}\|_{L^1_{2+\delta}}).\end{aligned}$$

Formula (2.4.18) then follows directly from (2.4.17). □

2.4.3 Proof of Theorem 2.2.2

Consider f_{in} satisfying (2.2.3) and such that $f_{in} \in L^1_{2s_0}(\mathbb{R}^3)$, for some $s_0 > 1$. There exists a sequence of functions $(f_{in,k})_{k\geq 1}$ in $\mathcal{C}^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap W^{3,\infty}(\mathbb{R}^3)$ such that $f_{in,k} \to f_{in}$ in $L^1_{2s_0}(\mathbb{R}^3)$ and

$$C'_k\, e^{-\delta'_k\, |v|^2} \leq f_{in,k} \leq \frac{C_k\, e^{-\delta_k\, |v|^2}}{1 + C_k\, e^{-\delta_k\, |v|^2}},$$

for some positive constants C_k, C'_k, δ_k and δ'_k.

For every $k \geq 1$, we set

$$\varepsilon_k = \frac{1}{k} \qquad \text{and} \qquad f_k = f^{\varepsilon_k},$$

where f^{ε_k} denotes the solution to (2.4.1), (2.4.2) with initial datum $f_{in,k}$ given by Theorem 2.4.2.

Lemma 2.4.15 *There are a non-negative function $f \in \mathcal{C}_w\big([0,T]; L^2(\mathbb{R}^3)\big) \cap L^\infty\big((0,T) \times \mathbb{R}^3\big)$ and a subsequence of $(f_k)_{k\geq 1}$ (not relabeled) which converges to f in $L^2\big(0,T; L^1(\mathbb{R}^3)\big)$, in $\mathcal{C}_w\big([0,T]; L^2(\mathbb{R}^3)\big)$ and a.e. on $(0,T) \times \mathbb{R}^3$.*

In addition, $0 \leq f \leq 1$ a.e. on $(0,T) \times \mathbb{R}^3$.

Here $\mathcal{C}_w\big([0,T]; L^2(\mathbb{R}^3)\big)$ denotes the space of weakly continuous functions in $L^2(\mathbb{R}^3)$. Since $0 \leq f_k \leq 1$, it follows from Lemma 2.4.15 and Hölder's inequality that $(f_k)_{k\geq 1}$ converges to f in $L^p\big((0,T) \times \mathbb{R}^3\big)$ for any $p \in [1, \infty[$.

Proof. For $m \geq 4$ and $r > 0$, we have

$$H^1_{2s_0-2-\gamma}(\mathbb{R}^3) \cap L^1_r(\mathbb{R}^3) \subset L^1(\mathbb{R}^3) \subset \left(H^m_{2+2\gamma}(\mathbb{R}^3)\right)',$$

the embedding of $H^1_{2s_0-2-\gamma}(\mathbb{R}^3) \cap L^1_r(\mathbb{R}^3)$ in $L^1(\mathbb{R}^3)$ being compact. Since the sequence $(f_{in,k})_{k\geq 1}$ converges to f_{in} in $L^1_{2s_0}(\mathbb{R}^3)$, there exists κ_0 such that $\|f_{in,k}\|_{L^1_{2s_0}} \leq \kappa_0$, and

$$\Gamma(\|f_{in,k}\|_{L^1_{2s_0}}) \leq \Gamma(\kappa_0), \qquad \text{for} \quad k \geq 1, \tag{2.4.19}$$

by Remark 2.4.12. We then deduce from Lemma 2.4.11 and Corollary 2.4.14 that

$$(f_k)_{k\geq 1} \text{ is bounded in } L^2\Big(0,T; H^1_{2s_0-2-\gamma}(\mathbb{R}^3)\cap L^1_2(\mathbb{R}^3)\Big). \tag{2.4.20}$$

Next, for $\varphi \in H^m_{2+2\gamma}(\mathbb{R}^3)$, we have

$$\begin{aligned}\int \partial_t f_k \varphi\, dv &= \sum_{i,j} \iint a^{\varepsilon_k}_{i,j}(v-v_*) f_k f_{k*}(1-f_{k*})\, \partial^2_{i,j}\varphi\, dv\, dv_* + \varepsilon_k \int f_k \Delta\varphi\, dv \\ &\quad + \sum_i \iint b^{\varepsilon_k}_i(v-v_*) f_k f_{k*}(1-f_k)\,(\partial_i\varphi - \partial_i\varphi_*)\, dv\, dv_*. \end{aligned}\tag{2.4.21}$$

Hence,

$$\begin{aligned}\left|\int \partial_t f_k\varphi\, dv\right| &\leq C\,\|\varphi\|_{W^{2,\infty}} \iint \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2 f_k f_{k*}(1-f_{k*}) dv\, dv_* \\ &\quad + C\,\|\varphi\|_{W^{2,\infty}}\,\|f_k\|^2_{L^1_2} + C\,\|\varphi\|_{H^2_{2+2\gamma}}\,\|f_k\|^{3/2}_{L^1_2} + \varepsilon_k\,\|\varphi\|_{W^{2,\infty}}\,\|f_{in}\|_{L^1}.\end{aligned}$$

Since $m\geq 4$, we infer from Lemma 2.4.11 and the continuous embedding of $H^m(\mathbb{R}^3)$ into $W^{2,\infty}(\mathbb{R}^3)$ that

$$(\partial_t f_k)_{k\geq 1} \text{ is bounded in } L^1\left(0,T; \left(H^m_{2+2\gamma}(\mathbb{R}^3)\right)'\right). \tag{2.4.22}$$

By [27, Corollary 4], we conclude from (2.4.20) and (2.4.22) that $(f_k)_{k\geq 1}$ is relatively compact in the space $L^2(0,T;L^1(\mathbb{R}^3))$. Therefore, there are a function $f \in L^2(0,T;L^1(\mathbb{R}^3))$ and a subsequence of $(f_k)_{k\geq1}$ (not relabeled) such that $(f_k)_{k\geq1}$ converges towards f in $L^2\big(0,T;L^1\big(\mathbb{R}^3\big)\big)$ and a.e. on $(0,T)\times\mathbb{R}^3$.

Moreover, we deduce from (2.4.21) that, for $\varphi\in\mathcal{C}^2_0(\mathbb{R}^3)$ with compact support included in B_R for some $R>0$, we have

$$\begin{aligned}&\left|\int f_k(t)\varphi\, dv - \int f_k(\sigma)\varphi\, dv\right| \\ &\quad\leq C\,\|\varphi\|_{W^{2,\infty}} \int_\sigma^t \iint_{|v|\leq R} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2\, f_k f_{k*}(1-f_{k*}) dv\, dv_*\, d\tau \\ &\quad\quad + C\,\|\varphi\|_{W^{2,\infty}}\,|t-\sigma|\left[\|f_k\|^2_{L^\infty(0,T;L^1_2)} + \|f_k\|^{3/2}_{L^\infty(0,T;L^1_2)} + \|f_{in}\|_{L^1}\right].\end{aligned}\tag{2.4.23}$$

From Lemma 2.4.11, we deduce that, for $R'>0$,

$$\int_0^T \iint_{|v|\leq R',\,|v_*|\leq R'} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^{2s_0}\, f_k f_{k*}(1-f_{k*})\, dv\, dv_*\, d\tau \leq \Gamma(\|f_{in,k}\|_{L^1_{2s_0}}) \leq \Gamma(\kappa_0).$$

We may then pass to the limit as $k\to+\infty$ thanks to the a.e. convergence of $(f_k)_{k\geq1}$ and $(\Psi_{\varepsilon_k})_{k\geq1}$ and then as $R'\to+\infty$ by the Fatou lemma to obtain

$$\int_0^T \iint |v-v_*|^\gamma\,|v_*|^{2s_0}\, f f_*(1-f_*)\, dv\, dv_*\, d\tau \leq \Gamma(\kappa_0). \tag{2.4.24}$$

Next, it is easy to check, by means of the a.e. convergence and (2.4.24), that

$$\left(\iint_{|v|\leq R} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2} f_k f_{k*}\,(1-f_{k*})\ |v_*|^2\, dv\, dv_*\right)_{k\geq 1}$$

converges towards

$$\iint_{|v|\leq R} |v-v_*|^\gamma\, |v_*|^2\, f f_*(1-f_*)\, dv\, dv_*$$

in $L^1(0,T)$. Therefore, the Vitali theorem implies that

$$\lim_{|t-\sigma|\to 0} \sup_{k\geq 1} \int_\sigma^t \iint_{|v|\leq R} f_k f_{k*}\,(1-f_{k*})\, \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2\, dv\, dv_*\, d\tau = 0.$$

We then deduce from (2.4.23) that the sequence $(\int f_k\varphi\, dv)_{k\geq 1}$ is equicontinuous and bounded in $\mathcal{C}([0,T])$. The Arzela-Ascoli theorem ensures that it is relatively compact in $\mathcal{C}([0,T])$. From the convergence of $(f_k)_{k\geq 1}$ towards f in $L^1\big((0,T)\times\mathbb{R}^3\big)$, we deduce that $\int f\,\varphi\, dv$ is the unique cluster point of $(\int f_k\varphi\, dv)_{k\geq 1}$. Therefore, $(\int f_k\varphi\, dv)_{k\geq 1}$ converges to $\int f\varphi\, dv$ in $\mathcal{C}([0,T])$. Since the sequence $(f_k)_{k\geq 1}$ and its limit f are bounded in $L^\infty(0,T;L^2(\mathbb{R}^3))$, it follows that $(f_k)_{k\geq 1}$ converges towards f in $\mathcal{C}_w([0,T];L^2(\mathbb{R}^3))$. □

Lemma 2.4.16 *The limit f of the sequence $(f_k)_{k\geq 1}$ is a solution to the Landau-Fermi-Dirac equation (2.2.1), (2.2.2) which satisfies (2.2.4) and (2.2.5).*

Proof. *Step 1: Conservation of mass and energy.*

Let $t\in[0,T]$. By Lemma 2.4.11 and (2.4.19), we have

$$\int_{|v|\leq R} f_k(t,v)\,|v|^{2s_0}\, dv \leq \int f_k(t,v)\,|v|^{2s_0}\, dv \leq \Gamma(\kappa_0),$$

for each $k\geq 1$. Thanks to Lemma 2.4.15 and the Fatou lemma, we may let $k\to+\infty$ and then $R\to+\infty$ and obtain

$$\int f(t,v)|v|^{2s_0} dv \leq \Gamma(\kappa_0). \tag{2.4.25}$$

Combining Lemma 2.4.11, Lemma 2.4.15 and (2.4.25), we see that $(M_{2r}(f_k))_{k\geq 1}$ converges strongly towards $M_{2r}(f)$ in $\mathcal{C}([0,T])$ for $r\in[0,s_0)$. Since $(f_{in,k})_{k\geq 1}$ converges to f_{in} in $L^1_{2s_0}(\mathbb{R}^3)$ and $s_0>1$, we deduce that

$$\begin{aligned}
\lim_{k\to+\infty} \int f_k(t,v)\, dv &= \lim_{k\to+\infty} \int f_{in,k}(v)\, dv = \int f_{in}(v)\, dv,\\
\lim_{k\to+\infty} \int |v|^2\, f_k(t,v)\, dv &= \lim_{k\to+\infty} \big(M_2(f_{in,k}) + 6\,\varepsilon_k\, M_0(f_{in,k})t\big) = M_2(f_{in}).
\end{aligned}$$

We thus conclude that f conserves mass and energy.

Step 2: Passage to the limit in the weak formulation (2.4.11).

For all $k \geq 1$, $\varphi \in \mathcal{C}_b^2(\mathbb{R}^3)$ and $t \in [0,T]$, the functions f_k satisfy,

$$\begin{aligned}
&\int f_k(t,v)\,\varphi(v)\,dv - \int f_{in,k}(v)\,\varphi(v)\,dv \\
&\qquad = \sum_{i,j} \int_0^t d\sigma \iint a_{i,j}^{\varepsilon_k}(v-v_*) f_k f_{k*}\,(1-f_{k*})\,\partial_{i,j}^2\varphi\,dv\,dv_* + \varepsilon_k \int_0^t d\sigma \int f_k \Delta\varphi\,dv \\
&\qquad\quad + \sum_i \int_0^t d\sigma \iint b_i^{\varepsilon_k}(v-v_*)\,f_k f_{k*}\,(2-f_k-f_{k*})\,\partial_i\varphi\,dv\,dv_*.
\end{aligned} \tag{2.4.26}$$

Our aim is here to pass to the limit as $k \to +\infty$ in formula (2.4.26). By Lemma 2.4.15, it is obvious for the left-hand side and the second integral in the right-hand side. We thus have to consider the two remaining integrals. As $(\Psi_{\varepsilon_k})_{k\geq 1}$ converges pointwise towards Ψ, the functions $a_{i,j}^{\varepsilon_k}$ and $b_i^{\varepsilon_k}$ defined at the beginning of Section 2.4.1 converge towards $a_{i,j}$ and b_i respectively. Consider $\varphi \in \mathcal{C}^2(\mathbb{R}^3)$ with compact support included in B_R for some $R > 0$. Let $R' > 0$. We first turn our attention to the integral involving the matrix a^{ε_k}.

$$\begin{aligned}
&\left| \sum_{i,j} \int_0^t d\sigma \iint \Big[a_{i,j}^{\varepsilon_k}(v-v_*) f_k f_{k*}\,(1-f_{k*}) - a_{i,j}(v-v_*) f f_*\,(1-f_*)\Big]\,\partial_{i,j}^2\varphi\,dv\,dv_* \right| \\
&\leq \left| \sum_{i,j} \int_0^t d\sigma \iint_{B_R\times B_{R'}} \Big[a_{i,j}^{\varepsilon_k}(v-v_*) f_k f_{k*}\,(1-f_{k*}) - a_{i,j}(v-v_*) f f_*\,(1-f_*)\Big]\,\partial_{i,j}^2\varphi\,dv\,dv_* \right| \\
&\quad + C\,\|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R,\,|v_*|\geq R'\}} \Psi_{\varepsilon_k}(|v-v_*|) f_k f_{k*}\,(1-f_{k*})\,dv\,dv_* \\
&\quad + C\,\|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R,\,|v_*|\geq R'\}} |v-v_*|^{\gamma+2} f f_*\,(1-f_*)\,dv\,dv_*.
\end{aligned} \tag{2.4.27}$$

The a.e. convergence of a^{ε_k} and f_k, the bound on f_k, the properties of Ψ_{ε_k} and the Lebesgue dominated convergence theorem imply that the first term of the right-hand side of (2.4.27) converges to zero. For the two others, it follows from (2.4.16) and (2.4.24) that

$$\begin{aligned}
&\int_0^t d\sigma \iint_{\{|v|\leq R,\,|v_*|\geq R'\}} |v-v_*|^{\gamma+2} f f_*\,(1-f_*)\,dv\,dv_* \\
&\qquad\qquad \leq 2\left[R^2 R'^{-2s_0} + R'^{2-2s_0}\right] \int_0^t d\sigma \iint |v-v_*|^{\gamma} |v_*|^{2s_0} f f_*\,(1-f_*)\,dv\,dv_* \\
&\qquad\qquad \leq 2\,\Gamma(\kappa_0)\left[R^2 R'^{-2s_0} + R'^{2-2s_0}\right],
\end{aligned}$$

and

$$\int_0^t d\sigma \iint_{\{|v|\leq R,\,|v_*|\geq R'\}} \Psi_{\varepsilon_k}(|v-v_*|) f_k f_{k*}\,(1-f_{k*})\,dv\,dv_* \leq 2\,\Gamma(\kappa_0)\big[R^2 R'^{-2s_0} + R'^{2-2s_0}\big].$$

We then substitute these estimates in (2.4.27) and let first $k \to +\infty$ and then $R' \to +\infty$ to obtain that the left-hand side converges to zero as $k \to +\infty$.

We proceed analogously for the integral of (2.4.26) which involves the function b^{ε_k}.

$$\begin{aligned}
&\left|\sum_i \int_0^t d\sigma \iint \Big[b_i^{\varepsilon_k}(v-v_*)\, f_k f_{k*}\,(2-f_k-f_{k*}) - b_i(v-v_*)\, f f_*\,(2-f-f_*)\Big]\,\partial_i\varphi\, dv\, dv_*\right| \\
&\qquad \le \left|\sum_i \int_0^t d\sigma \iint_{B_R\times B_{R'}} \Big[b_i^{\varepsilon_k}\, f_k f_{k*}\,(2-f_k-f_{k*}) - b_i\, f f_*\,(2-f-f_*)\Big]\,\partial_i\varphi\, dv\, dv_*\right| \\
&\qquad\quad + C\,\|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\le R,\,|v_*|\ge R'\}} \frac{\Psi_{\varepsilon_k}(v-v_*)}{|v-v_*|}\, f_k f_{k*}\, dv\, dv_* \\
&\qquad\quad + C\,\|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\le R,\,|v_*|\ge R'\}} |v-v_*|^{1+\gamma}\, f f_*\, dv\, dv_*. \qquad (2.4.28)
\end{aligned}$$

For the first term of the right-hand side, we use again the a.e. convergence of a^{ε_k} and f_k, the bound on f_k, the properties of Ψ_{ε_k} and the Lebesgue dominated convergence theorem, whereas for the two others, we have

$$\begin{aligned}
&\int_0^t d\sigma \iint_{\{|v|\le R,\,|v_*|\ge R'\}} |v-v_*|^{1+\gamma}\, f f_*\, dv\, dv_* \\
&\qquad \le C \int_0^t d\sigma \iint_{\{|v|\le R,\,|v_*|\ge R'\}} \left(1+|v|^2\right)^{(1+\gamma)/2} \left(1+|v_*|^2\right)^{(1+\gamma)/2} f f_*\, dv\, dv_* \\
&\qquad \le C\, T\, (1+R'^2)^{(1+\gamma-2s_0)/2}\, \|f\|_{L^1_2}\, \|f\|_{L^1_{2s_0}},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t d\sigma \iint_{\{|v|\le R,|v_*|\ge R'\}} \frac{\Psi_{\varepsilon_k}(v-v_*)}{|v-v_*|}\, f_k f_{k*}\, dv\, dv_* \;&\le\; C\, T\, (1+R'^2)^{(1+\gamma-2s_0)/2}\, \|f_k\|^2_{L^1_{2s_0}} \\
&\le\; C\, T\, (1+R'^2)^{(1+\gamma-2s_0)/2}\, \Gamma(\kappa_0)^2,
\end{aligned}$$

by Lemma 2.4.11. Inserting the estimates in (2.4.28) and letting first $k \to +\infty$ and then $R' \to +\infty$, we obtain that the left-hand side converges to zero as $k \to +\infty$.

Therefore, f is a weak solution to the Landau-Fermi-Dirac equation (2.2.1), (2.2.2) which preserves mass and energy. □

Moreover, we deduce from (2.4.24) and (2.4.25) that f satisfies

$$f(1-f) \in L^1_{loc}\big(\mathbb{R}_+; L^1_{2s_0+\gamma}(\mathbb{R}^3)\big) \quad \text{and} \quad f \in L^\infty_{loc}\big(\mathbb{R}_+; L^1_{2s_0}(\mathbb{R}^3)\big). \qquad (2.4.29)$$

Distinguishing the cases $s_0 < 1+\gamma/2$ and $s_0 \ge 1+\gamma/2$, we infer from (2.4.17) and (2.4.18) the existence of a constant $C(T,\kappa_0)$ such that, for all $R>0$, $k \ge R/2$,

$$\left(\frac{1}{2}\right)^{2+\gamma} \int_0^T \int_{|v|\le R} |\nabla f_k|^2 \left(1+|v|^2\right)^{s_0} dv\, d\tau \le (1+\varepsilon_k)\, C(T,\kappa_0).$$

Letting first $k \to +\infty$ thanks to a weak compactness argument and then $R \to +\infty$ by the Fatou lemma, we conclude that

$$\nabla f \in L^2_{loc}\big(\mathbb{R}_+; L^2_{2s_0}(\mathbb{R}^3)\big). \tag{2.4.30}$$

Therefore, the proof of the first statement of Theorem 2.2.2 is now complete.

We now verify that the entropy of f is a non-decreasing function when $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$, which corresponds to the second statement of Theorem 2.2.2. For that purpose, we first need a smoothness result.

Lemma 2.4.17 *Let $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$ satisfying (2.2.3). The weak solution f to (2.2.1), (2.2.2) given by Lemma 2.4.16 belongs to $\mathcal{C}([0,T]; L^2(\mathbb{R}^3))$.*

Proof. Let us first show that

$$\partial_t f \in L^2\Big(0,T; \big(H^1_{2+\gamma}(\mathbb{R}^3)\big)'\Big). \tag{2.4.31}$$

Indeed, the function f satisfies, in the sense of distributions,

$$\partial_t f = \nabla \cdot \Big[\overline{A}\nabla f - \overline{b} f(1-f)\Big].$$

Moreover, since the initial datum belongs to $L^1_{2+\gamma}(\mathbb{R}^3)$, we have

$$f \in L^\infty\big(0,T; L^1_{2+\gamma}(\mathbb{R}^3)\big) \cap L^2\big(0,T; H^1_{2+\gamma}(\mathbb{R}^3)\big),$$

by (2.4.29) and (2.4.30). Consequently,

$$\begin{aligned} \|\nabla \cdot \big[\overline{A}\nabla f\big]\|_{(H^1_{2+\gamma})'} &\leq C\|f\|_{L^1_{2+\gamma}}\|f\|_{H^1_{2+\gamma}}, \\ \|\nabla \cdot \big[\overline{b} f(1-f)\big]\|_{(H^1_{2+\gamma})'} &\leq C\|f\|^2_{L^1_2}, \end{aligned}$$

whence (2.4.31). Since

$$H^1_{2+\gamma}(\mathbb{R}^3) \subset L^2_{2+\gamma}(\mathbb{R}^3) \subset (H^1_{2+\gamma}(\mathbb{R}^3))',$$

with continuous and dense embeddings, and

$$f \in L^2(0,T; H^1_{2+\gamma}(\mathbb{R}^3)) \quad \text{and} \quad \partial_t f \in L^2(0,T; (H^1_{2+\gamma}(\mathbb{R}^3))'),$$

we have $f \in \mathcal{C}([0,T]; L^2_{2+\gamma}(\mathbb{R}^3))$ by [21, Proposition 1.2.1 and Theorem 1.3.1] (see also [14, Theorem 5.9.3]). Lemma 2.4.17 then follows since $L^2_{2+\gamma}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$. □

Lemma 2.4.18 *Let $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$ satisfying (2.2.3). Let f denote the weak solution to (2.2.1), (2.2.2) given by Lemma 2.4.16. The entropy $S(f)$ is a continuous and non-decreasing function such that, for $t \geq 0$,*

$$S(f_{in}) \leq S(f)(t) \leq E_{in} + \int e^{-|v|^2}\, dv. \tag{2.4.32}$$

Proof. We first show the continuity of $S(f)$. Let $t \geq 0$ and $(t_n)_{n\geq 1}$ be a sequence converging to t. Lemma 2.4.17 implies that $(f(t_n))_{n\geq 1}$ converges towards $f(t)$ in $L^2(\mathbb{R}^3)$. One can extract a subsequence $f(t_{\varphi(n)})_{n\geq 1}$ which converges a.e. in $\mathbb{R}^3$ towards $f(t)$.

From the inequality

$$s(r) \leq r|v|^2 + e^{-|v|^2} \qquad \text{for } 0 \leq r \leq 1, \tag{2.4.33}$$

where $s(r) = r|\ln r| + (1-r)|\ln(1-r)|$, we deduce that

$$\begin{aligned}
&\Big|S(f)(t_{\varphi(n)}) - S(f)(t)\Big| \\
&\leq \left|\int_{|v|\leq R} \Big(s(f)(t_{\varphi(n)}) - s(f)(t)\Big)\,dv\right| + \int_{|v|\geq R} \Big(f(t_{\varphi(n)}) + f(t)\Big)|v|^2 dv + 2\int_{|v|\geq R} e^{-|v|^2} dv \\
&\leq \left|\int_{|v|\leq R} \Big(s(f)(t_{\varphi(n)}) - s(f)(t)\Big)\,dv\right| + 2R^{-\gamma}\,\Gamma(\kappa_0) + 2\int_{|v|\geq R} e^{-|v|^2} dv,
\end{aligned}$$

hence the convergence of $(S(f)(t_{\varphi(n)}))_{n\geq 1}$ towards $S(f)(t)$. Since $(S(f)(t_n))_{n\geq 1}$ is bounded by (2.2.5) and (2.4.33) and has a unique cluster point $S(f)(t)$, we conclude that $(S(f)(t_n))_{n\geq 1}$ converges to $S(f)(t)$.

Let us now prove the monotonicity of $S(f)$. Consider $h \geq 0$ and $0 \leq \sigma \leq t$. We deduce from (2.4.15) that

$$hS(f_{in,k}) \leq \int_\sigma^{\sigma+h} S(f_k)(\tau)\,d\tau \leq \int_t^{t+h} S(f_k)(\tau)\,d\tau. \tag{2.4.34}$$

As previously, (2.4.33) imply that

$$\int_0^T \Big|S(f_k) - S(f)\Big|\,dt \leq \int_0^T \left|\int_{B_R}\Big(s(f_k) - s(f)\Big)\,dv\right| dt + 2TR^{-\gamma}\,\Gamma(\kappa_0) + 2T\int_{|v|\geq R} e^{-|v|^2} dv,$$

and thus that $(S(f_k))_{k\geq 1}$ converges to $S(f)$ in $L^1(0,T)$. Similarly, $(S(f_{in,k}))_{k\geq 1}$ converges to $S(f_{in})$. We may then pass to the limit as $k \to +\infty$ in (2.4.34) to obtain

$$S(f_{in}) \leq \frac{1}{h}\int_\sigma^{\sigma+h} S(f)(\tau)\,d\tau \leq \frac{1}{h}\int_t^{t+h} S(f)(\tau)\,d\tau.$$

Letting $h \to 0$ thanks to the continuity of $S(f)$ completes the proof of the monotonicity of $S(f)$ and the first inequality in (2.4.32). Finally, the second inequality in (2.4.32) follows from (2.4.33). □

2.5 Uniqueness

In this section, we are concerned with the uniqueness issue. As previously mentioned, we first need an embedding lemma for weighted Sobolev spaces because of the non-quadratic nature of the LFD collision operator.

Lemma 2.5.1 *For all $r \geq 0$, $\varepsilon > 0$, there exists a constant $C > 0$ such that, for every function $h \in H^1_{2r}(\mathbb{R}^3)$, we have*

$$\|h\|_{L^4_{2r}} \leq C\varepsilon^{-3/4}\|h\|_{L^2_{2r}} + C\varepsilon^{1/4}\|\nabla h\|_{L^2_{2r}}.$$

The proof of Lemma 2.5.1 is an easy extension of [26, Lemma 3.6.7] where the above inequality is established for $r = 0$.

Theorem 2.5.2 *Let $f_{in} \in L^1_{2s}(\mathbb{R}^3)$ with $2s > 4\gamma + 11$, satisfying (2.2.3). Then there is a unique weak solution f to (2.2.1), (2.2.2) (in the sense of Definition 2.2.1) such that*

$$f \in L^\infty_{loc}(\mathbb{R}_+; L^2_{2s}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}_+; H^1_{2s}(\mathbb{R}^3)).$$

Remark 2.5.3 *Since $0 \leq f_{in} \leq 1$, f_{in} belongs to $L^2_{2s}(\mathbb{R}^3)$ as soon as it belongs to $L^1_{2s}(\mathbb{R}^3)$. Thus we do not need any extra assumption in a weighted L^2-space as in [10].*

Proof. We only give formal computations in order to highlight the difference with the proof for the classical Landau equation performed in [10, Theorem 7]. Let f_1 and f_2 be two solutions to (2.2.1), (2.2.2) satisfying the requirements of Theorem 2.5.2. We set $u = f_1 - f_2$ and $w = f_1 + f_2$. The function u satisfies, in the sense of distributions,

$$\partial_t u = \frac{1}{2}\nabla \cdot \Big\{\overline{A}^{u(1-w)}\,\nabla w + \big(\overline{A}^{f_1} + \overline{A}^{f_2}\big)\,\nabla u - \overline{b}^{u}\,\big[f_1(1-f_1) + f_2(1-f_2)\big] - \overline{b}^{w}\,u\,(1-w)\Big\}.$$

Then, for every $q > 0$,

$$\begin{aligned}
\frac{d}{dt}\int |u|^2(1+|v|^2)^q\,dv \;=\; & -\int \overline{A}^{u(1-w)}\nabla w\,\nabla\big[u(1+|v|^2)^q\big]dv && (2.5.1)\\
& -\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla u\,\nabla\big[u(1+|v|^2)^q\big]dv && (2.5.2)\\
& +\int \overline{b}^{u}\big[f_1(1-f_1) + f_2(1-f_2)\big]\,\nabla\big[u(1+|v|^2)^q\big]dv && (2.5.3)\\
& +\int \overline{b}^{w}u\,(1-w)\,\nabla\big[u(1+|v|^2)^q\big]dv. && (2.5.4)
\end{aligned}$$

We first consider (2.5.2) and (2.5.4),

$$\begin{aligned}
(2.5.2) + (2.5.4) \;=\; & -q\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla\big(u^2\big)\big(1+|v|^2\big)^{q-1}v\,dv\\
& -\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla u\nabla u\,\big(1+|v|^2\big)^q\,dv + \frac{1}{2}\int \overline{b}^{w}\cdot\nabla\big(u^2\big)\big(1+|v|^2\big)^q\,dv\\
& -\int u\,w\,\overline{b}^{w}\cdot\nabla u\,(1+|v|^2)^q dv + 2q\int u^2\,(1-w)\,\overline{b}^{w}\cdot v\,\big(1+|v|^2\big)^{q-1}dv.
\end{aligned}$$

With the ellipticity of the diffusion matrix and an integration by parts in the integrals involving the term $\nabla\big(u^2\big)$, we find

$$\begin{aligned}
&(2.5.2) + (2.5.4)\\
&\quad \le -2K\int |\nabla u|^2\left(1+|v|^2\right)^{q+\gamma/2}dv + q\int u^2\left[\overline{B}^{f_1}+\overline{B}^{f_2}\right]\cdot v\left(1+|v|^2\right)^{q-1}dv\\
&\qquad + q\int u^2\Big[\overline{A}^{f_1}+\overline{A}^{f_2}\Big] : \nabla\big[\big(1+|v|^2\big)^{q-1}v\big]dv - \frac{1}{2}\int u^2\,\overline{c}^w\big(1+|v|^2\big)^q dv\\
&\qquad - q\int u^2\,\overline{b}^w\cdot v\left(1+|v|^2\right)^{q-1}dv - \int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q\,dv\\
&\qquad + 2q\int u^2\,\overline{b}^w\cdot v\left(1+|v|^2\right)^{q-1}dv - 2q\int u^2\,w\,\overline{b}^w\cdot v\left(1+|v|^2\right)^{q-1}dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
(2.5.2)+(2.5.4) \;\le\; & -2K\int |\nabla u|^2\left(1+|v|^2\right)^{q+\gamma/2}dv + \int u^2\,E\,dv\\
& + \left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q dv\right| + 2q\left|\int u^2\,w\,\overline{b}^w\cdot v\left(1+|v|^2\right)^{q-1}dv\right|,
\end{aligned}$$

where

$$E = q\Big[\overline{B}^{f_1}+\overline{B}^{f_2}+\overline{b}^w\Big]\cdot v\left(1+|v|^2\right)^{q-1} + q\Big[\overline{A}^{f_1}+\overline{A}^{f_2}\Big] : \nabla\big[\big(1+|v|^2\big)^{q-1}v\big] - \frac{1}{2}\,\overline{c}^w\big(1+|v|^2\big)^q.$$

Now,

$$\begin{aligned}
E \;=\; & q\int \big[f_{1*}(1-f_{1*})+f_{2*}(1-f_{2*})\big]|v-v_*|^\gamma\big(1+|v|^2\big)^{q-2}\\
&\times\Big\{-2|v|^2(v\cdot v_*)+2q|v|^2|v_*|^2-2(q-1)(v\cdot v_*)^2-2\,v\cdot v_*+2\,|v_*|^2\Big\}\,dv_*\\
&+\int w_*|v-v_*|^\gamma\big(1+|v|^2\big)^{q-1}\Big\{(\gamma+3-2q)|v|^2+2\,q\,(v\cdot v_*)+\gamma+3\Big\}\,dv_*,
\end{aligned}$$

and choosing $2q > \gamma+3$, we deduce that $E \le C\big(1+|v|^2\big)^q$ since $f_i \in L^\infty_{loc}(\mathbb{R}_+; L^1_{\gamma+2}(\mathbb{R}^3))$ for $i = 1, 2$. Consequently,

$$\begin{aligned}
(2.5.2)+(2.5.4) \le & -2K\int |\nabla u|^2\left(1+|v|^2\right)^{q+\gamma/2}dv + C\int u^2\left(1+|v|^2\right)^q dv\\
& + \left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q dv\right| + 2q\left|\int u^2\,w\,\overline{b}^w\cdot v\left(1+|v|^2\right)^{q-1}dv\right|.
\end{aligned} \tag{2.5.5}$$

From Hölder's inequality and Lemma 2.5.1, we deduce that for every $\varepsilon > 0$,

$$\begin{aligned}
\left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q\,dv\right| \;\le\; & C\|w\|^{1/2}_{L^2_{2q+2\gamma+4}}\|\nabla u\|_{L^2_{2q+\gamma}}\|u\|_{L^4_{2q}}\\
\le\; & C_\varepsilon\|w\|^4_{L^2_{2q+2\gamma+4}}\|u\|^2_{L^2_{2q}} + \varepsilon\,\|\nabla u\|^2_{L^2_{2q+\gamma}},
\end{aligned} \tag{2.5.6}$$

and

$$\left|\int u^2\, w\, \overline{b}^w \cdot v \left(1+|v|^2\right)^{q-1} dv\right| \le C\|w\|_{L^2_{2q+2\gamma}} \|u\|^2_{L^4_{2q}}$$
$$\le C_\varepsilon \|w\|^4_{L^2_{2q+2\gamma}} \|u\|^2_{L^2_{2q}} + \varepsilon \,\|\nabla u\|^2_{L^2_{2q+\gamma}}. \qquad (2.5.7)$$

Finally, substituting (2.5.6) and (2.5.7) in (2.5.5), we find

$$(2.5.2) + (2.5.4) \le -2\,(K-\varepsilon)\,\|\nabla u\|^2_{L^2_{2q+\gamma}} + C_\varepsilon \|u\|^2_{L^2_{2q}} \left(1 + \|w\|^4_{L^2_{2q+2\gamma+4}}\right). \qquad (2.5.8)$$

It remains now to consider (2.5.1) and (2.5.3). In the sequel, we use the notation Π for $\Pi(v-v_*)$. We have

$$(2.5.1) + (2.5.3) = -\iint \Pi|v-v_*|^{\gamma+2} u_*(1-w_*)\nabla w \nabla u\,(1+|v|^2)^q\, dv\, dv_* \qquad (2.5.9)$$
$$-2\,q \iint \Pi|v-v_*|^{\gamma+2} u_*(1-w_*)\nabla w\; u\; v\,(1+|v|^2)^{q-1}\, dv\, dv_* \qquad (2.5.10)$$
$$-2 \iint (v-v_*)\cdot \nabla u\, |v-v_*|^{\gamma} u_* \big[f_1(1-f_1) + f_2(1-f_2)\big]\,(1+|v|^2)^q\, dv\, dv_* \qquad (2.5.11)$$
$$-4\,q \iint (v-v_*)\cdot v\, |v-v_*|^{\gamma} u_* \big[f_1(1-f_1) + f_2(1-f_2)\big] u\,(1+|v|^2)^{q-1}\, dv\, dv_*. \qquad (2.5.12)$$

Using successively the Cauchy-Schwarz inequality, $|v-v_*|^{2r} \le C\big(1+|v|^2\big)^r\big(1+|v_*|^2\big)^r$ and the Fubini theorem, we find

$$(2.5.9) \le C \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|\nabla u|^2 \left(1+|v|^2\right)^{q-1} dv\, dv_*\right\}^{1/2}$$
$$\times \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|\nabla w|^2 \left(1+|v|^2\right)^{q+1} dv\, dv_*\right\}^{1/2}$$
$$\le C\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}}\,\|\nabla w\|_{L^2_{2q+\gamma+4}},$$

$$(2.5.10) \le C \left\{\iint |v-v_*|^{\gamma+2}\,|v|^2\,|u_*|\,|\nabla w|^2 \left(1+|v|^2\right)^{q-1+\gamma/2} dv\, dv_*\right\}^{1/2}$$
$$\times \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|u|^2 \left(1+|v|^2\right)^{q-1-\gamma/2} dv\, dv_*\right\}^{1/2}$$
$$\le C\,\|u\|_{L^1_{\gamma+2}}\,\|u\|_{L^2_{2q}}\,\|\nabla w\|_{L^2_{2q+2\gamma+2}},$$

$$(2.5.11) \le C \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|w|^2 \left(1+|v|^2\right)^{q} dv\, dv_*\right\}^{1/2}$$
$$\times \left\{\iint |v-v_*|^{\gamma}\,|u_*|\,|\nabla u|^2 \left(1+|v|^2\right)^{q} dv\, dv_*\right\}^{1/2}$$
$$\le C\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}}\,\|w\|_{L^2_{2q+\gamma+2}},$$

$$
\begin{aligned}
(2.5.12) \quad &\leq C \left\{ \iint |v - v_*|^{\gamma+2} \, |u_*| \, |w|^2 \left(1 + |v|^2\right)^{q-1+\gamma/2} dv\, dv_* \right\}^{1/2} \\
&\quad \times \left\{ \iint |v - v_*|^{\gamma} \, |v|^2 \, |u_*| \, |u|^2 \left(1 + |v|^2\right)^{q-1-\gamma/2} dv\, dv_* \right\}^{1/2} \\
&\leq C \, \|u\|_{L^1_{\gamma+2}} \, \|u\|_{L^2_{2q}} \, \|w\|_{L^2_{2q+2\gamma}}.
\end{aligned}
$$

Since $\gamma \leq 1$, we thus obtain

$$
\begin{aligned}
(2.5.1) + (2.5.3) \quad &\leq C\, B(t) \, \|u\|_{L^1_{\gamma+2}} \, \|\nabla u\|_{L^2_{2q+\gamma}} + C\, B(t) \, \|u\|_{L^1_{\gamma+2}} \, \|u\|_{L^2_{2q}} \\
&\quad + C\, A \, \|u\|_{L^1_{\gamma+2}} \, \|\nabla u\|_{L^2_{2q+\gamma}} + C\, A \, \|u\|_{L^1_{\gamma+2}} \, \|u\|_{L^2_{2q}}, \\
&\leq C \left(A + B(t)\right) \|u\|_{L^1_{\gamma+2}} \left(\|\nabla u\|_{L^2_{2q+\gamma}} + \|u\|_{L^2_{2q}} \right),
\end{aligned}
$$

where $A = \sup_{t\in[0,T]} \|w(t)\|_{L^2_{2q+2\gamma+4}}$ and $B(t) = \|\nabla w(t)\|_{L^2_{2q+\gamma+4}}$.

Now, for $\delta > 0$, we have $\|u\|_{L^1_{\gamma+2}} \leq C_\delta \|u\|_{L^2_{2\gamma+7+\delta}}$, and thus, for $2q > 2\gamma + 7$,

$$
(2.5.1) + (2.5.3) \leq \varepsilon \, \|\nabla u\|^2_{L^2_{2q+\gamma}} + C_\varepsilon \left(1 + A^2 + B^2(t)\right) \|u\|^2_{L^2_{2q}}. \tag{2.5.13}
$$

From (2.5.8) and (2.5.13), we infer that

$$
\|u\|^2_{L^2_{2q}}(t) \leq C \int_0^t \left(1 + A^2 + A^4 + B^2(\tau)\right) \, \|u\|^2_{L^2_{2q}}(\tau)\, d\tau.
$$

Since A is finite and B belongs to $L^2_{loc}(\mathbb{R}_+)$, we may use the Gronwall lemma and conclude that $u = 0 = f_1 - f_2$. □

Appendix: Well-posedness of (2.4.3)

We give here further details for the proof of the well-posedness statement of Theorem 2.4.3. In order to apply [17, Theorem 5.8.1], we introduce the quasi-linear problem

$$
\begin{aligned}
\partial_t f &= \nabla \cdot \left(\left(\overline{A}^{g,\varepsilon} + \varepsilon I_3\right) \nabla f \right) - (1 - 2f)\, \overline{b}^{g,\varepsilon} \cdot \nabla f - \overline{c}^{g,\varepsilon} f\, \theta(f) && (2.5.14) \\
f(0,.) &= f_{in}. && (2.5.15)
\end{aligned}
$$

where the function θ is defined on $\mathbb{R}$ by

$$
\theta(f) = \begin{cases} 1 & \text{if} \quad f \leq 0 \\ 1 - f & \text{if} \quad 0 \leq f \leq 1 \\ 0 & \text{if} \quad f \geq 1 \end{cases}
$$

Let $\delta \in (0, 1)$. Then, $f_{in} \in \mathcal{H}^{2+\delta}\left(\mathbb{R}^3\right)$. Owing to Lemma 2.4.4 and the uniform ellipticity (2.4.7), the functions α_i and α defined by

$$
\alpha_i(t, v, \xi) = \sum_j \left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon \delta_{i,j}\right) \xi_j \quad \text{and} \quad \alpha(t, v, f, \xi) = (1 - 2f) \sum_k \overline{b}^{g,\varepsilon}_k \xi_k + \overline{c}^{g,\varepsilon} f\, \theta(f)
$$

satisfy the assumptions of [17, Theorem 5.8.1], which implies the existence of a solution f^ε to (2.5.14), (2.5.15) belonging to the Hölder space $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$. Moreover, there exists a constant Λ depending only on f_{in}, δ, T, ε and C_L such that

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,(2+\delta)/2}} \leq \Lambda.$$

It remains to prove that $0 \leq f^\varepsilon(t,v) \leq 1$. To this aim, we consider the linear operator $\mathcal{L}_1$ defined by

$$\mathcal{L}_1 u = \partial_t u - \sum_{i,j} \left(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\,\delta_{i,j}\right) \partial^2_{i,j} u - \sum_i \left[\overline{B}_i^{g,\varepsilon} - \overline{b}_i^{g,\varepsilon}(1-2f^\varepsilon)\right]\partial_i u + \overline{c}^{g,\varepsilon}\,\theta(f^\varepsilon)\,u.$$

Let $R>0$. As soon as

$$C \geq 6K_\varepsilon\|f_{in}\|_{L^1} + 6\varepsilon + 12\,(1+\Lambda)^2 K_\varepsilon^2\|f_{in}\|^2_{L^1} \quad \text{and} \quad \lambda \geq 1 + K_\varepsilon\|f_{in}\|_{L^1},$$

we deduce from the comparison principle [17, Theorem 1.2.1] that

$$f^\varepsilon(t,v) \geq -\frac{\Lambda}{R^2}\left(|v|^2 + Ct\right)e^{\lambda t}, \quad (t,v)\in[0,T]\times B_R.$$

We let $R\to+\infty$ and obtain $f^\varepsilon(t,v)\geq 0$ for every $(t,v)\in[0,T]\times\mathbb{R}^3$.

Next, we introduce the quasi-linear operator $\mathcal{L}_2$ defined by

$$\mathcal{L}_2 u = \partial_t u - \sum_{i,j} \left(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\,\delta_{i,j}\right) \partial^2_{i,j} u - \sum_i \left[\overline{B}_i^{g,\varepsilon} - \overline{b}_i^{g,\varepsilon}(1-2f^\varepsilon)\right]\partial_i u + \overline{c}^{g,\varepsilon} f^\varepsilon\,\theta(u).$$

Let $R>0$. For

$$C \geq 6K_\varepsilon\|f_{in}\|_{L^1} + 6\varepsilon + 12\,(1+\Lambda)^2 K_\varepsilon^2\|f_{in}\|^2_{L^1},$$

it follows from the comparison principle for quasi-linear equations [19, Theorem 9.1] that

$$f^\varepsilon(t,v) \leq 1 + \frac{\Lambda}{R^2}\left(|v|^2 + Ct\right)e^{t}, \qquad (t,v)\in[0,T]\times B_R.$$

Letting R go to infinity, we obtain that $f^\varepsilon(t,v)\leq 1$ for every $(t,v)\in[0,T]\times\mathbb{R}^3$.

Consequently, there exists a solution to (2.4.4), (2.4.5) in $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$. The uniqueness of such a solution follows easily from the comparison principle [17, Theorem 1.2.5].

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Annexe A

Existence pour le problème approché (2.4.4)-(2.4.5)

A.1 Le problème approché

Commençons par rappeler la définition du problème approché considéré dans le Chapitre 2. Soit $(\Psi_\varepsilon)_{\varepsilon>0}$ une famille de fonctions bornées, de classe $\mathcal{C}^\infty$ sur $\mathbb{R}_+$ qui coïncident avec $\Psi(r) = r^{\gamma+2}$ pour $0 < \varepsilon < r < \varepsilon^{-1}$ et vérifient

- (i) $\Psi_\varepsilon \in W^{4,\infty}(\mathbb{R}_+)$
- (ii) Pour $0 < r < \varepsilon^{-1}$, $\Psi_\varepsilon(r) \geq r^{\gamma+2}/2$; Pour $r > \varepsilon^{-1}$, $\Psi_\varepsilon(r) \geq \varepsilon^{-(\gamma+2)}/2 > 0$;
- (iii) Pour tout $r > 0$, $\Psi_\varepsilon(r) \leq r^2(1 + r^\gamma)$ et $|\Psi_\varepsilon'(r)| \leq (\gamma + 2)\, r\,(1 + r^\gamma)$;
- (iv) Pour $0 < r < \varepsilon$, $\Psi_\varepsilon(r) = r^2\,\nu_\varepsilon(r)$, avec $\nu_\varepsilon \in \mathcal{C}^\infty([0, \varepsilon])$, $\nu_\varepsilon(0) = 1$, $\nu_\varepsilon'(0) = 0$ et $\nu_\varepsilon''(0) = 0$.

On note alors pour $(i, j) \in [\![1, 3]\!]^2$,

$$a^\varepsilon_{i,j}(z) = \Psi_\varepsilon(|z|)\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right), \qquad b^\varepsilon_i(z) = \sum_k \partial_k a^\varepsilon_{i,k}(z) = -\,\frac{2\,z_i}{|z|^2}\,\Psi_\varepsilon(|z|)$$

$$\text{et} \qquad c^\varepsilon(z) = \sum_{k,l} \partial^2_{kl} a^\varepsilon_{k,l}(z) = -\,\frac{2}{|z|^2}\left[\Psi_\varepsilon(|z|) + |z|\,\Psi_\varepsilon'(|z|)\right].$$

On définit également

$$\overline{b}^{f,\varepsilon} = b^\varepsilon * f, \quad \overline{c}^{f,\varepsilon} = c^\varepsilon * f, \quad \overline{A}^{f,\varepsilon} = a^\varepsilon * (f(1-f)) \quad \text{et} \quad \overline{B}^{f,\varepsilon} = b^\varepsilon * (f(1-f)).$$

On considère alors le problème suivant

$$\partial_t f = \nabla \cdot \left[\left(\overline{A}^{g,\varepsilon} + \varepsilon I_3\right)\nabla f - \overline{b}^{g,\varepsilon} f(1-f)\right], \tag{A.1.1}$$

$$f(0,.) = f_{in}, \tag{A.1.2}$$

où $f_{in} \in \mathcal{C}^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap W^{3,\infty}(\mathbb{R}^3)$ est telle que, pour des constantes strictement positives α_1, α_2, β_1 et β_2, on ait, pour tout v de $\mathbb{R}^3$,

$$0 < \alpha_1\, e^{-\beta_1 |v|^2} \le f_{in}(v) \le \frac{\alpha_2\, e^{-\beta_2|v|^2}}{1+\alpha_2\, e^{-\beta_2|v|^2}} < 1, \tag{A.1.3}$$

et où la fonction g appartient à l'ensemble $\mathcal{C}$ défini par

$$\mathcal{C} = \left\{ g \in \mathcal{C}\big([0,T]; L^1(\mathbb{R}^3)\big);\ \begin{array}{c} 0 \le g \le 1, \qquad \forall\, s,t \in [0,T], \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}^3), \\ \displaystyle\int g(t,v)\,dv = \int f_{in}(v)\,dv \\ \displaystyle\left| \int \big(g(t,v) - g(s,v)\big)\, \varphi(v)\, dv \right| \le C_L\, \|\varphi\|_{\mathcal{C}_b^2(\mathbb{R}^3)}\, |t-s| \\ \displaystyle\left| \int \big(g(1-g)(t,v) - g(1-g)(s,v)\big)\, \varphi(v)\, dv \right| \\ \le C_L\, \|\varphi\|_{\mathcal{C}_b^2(\mathbb{R}^3)}\, |t-s| \\ \displaystyle\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le g(t,v) \le \frac{\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}{1+\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}} \end{array} \right\}.$$

Les valeurs des constantes $\beta_1' \ge \beta_1$, D, E, F et C_L seront précisées ultérieurement.

Dans la Section A.2, on montre qu'il existe une unique solution de (A.1.1)-(A.1.2). Ensuite, dans la Section A.3, on prouve que, pour un choix convenable des constantes β_1', D, E, F et C_L ne dépendant que de f_{in}, T et ε, cette solution appartient à l'ensemble $\mathcal{C}$.

A.2 Existence et unicité

Dans cette section, on montre la proposition suivante :

Proposition A.2.1 *Soient $\delta \in (0,1)$ et $\varepsilon > 0$. Pour tout élément g de $\mathcal{C}$, il existe une unique solution classique f^ε du problème (A.1.1)-(A.1.2) appartenant à l'espace de Hölder $\mathcal{H}^{2+\delta,\,(2+\delta)/2}\big([0,T]\times\mathbb{R}^3\big)$ et on a*

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,(2+\delta)/2}} \le \Lambda. \tag{A.2.1}$$

où la constante Λ ne dépend que de f_{in}, δ, T, ε et C_L.

Preuve. On considère le problème quasi-linéaire

$$\partial_t f = \nabla \cdot \Big(\big(\overline{A}^{g,\varepsilon} + \varepsilon I_3\big) \nabla f \Big) - (1-2f)\, \overline{b}^{g,\varepsilon} \cdot \nabla f - \overline{c}^{g,\varepsilon} f\, \theta(f) \tag{A.2.2}$$

$$f(0,.) = f_{in}. \tag{A.2.3}$$

où la fonction θ est définie sur $\mathbb{R}$ par

$$\theta(f) = \begin{cases} 1 & \text{si} \quad f \le 0 \\ 1-f & \text{si} \quad 0 \le f \le 1 \\ 0 & \text{si} \quad f \ge 1 \end{cases}$$

Soit $\delta \in (0,1)$. On note qu'avec les hypothèses faites sur la condition initiale f_{in}, on a $f_{in} \in \mathcal{H}^{2+\delta}\big(\mathbb{R}^3\big)$. De plus, les fonctions α_i et α définies par

$$\alpha_i(t,v,\xi) = \sum_j (\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\delta_{i,j})\,\xi_j \quad \text{et} \quad \alpha(t,v,f,\xi) = (1-2f)\sum_k \overline{b}^{g,\varepsilon}_k \xi_k + \overline{c}^{g,\varepsilon} f\,\theta(f)$$

sont continues. Les fonctions α_i sont différentiables par rapport à v et ξ. Comme les fonctions $\overline{A}^{g,\varepsilon}_{i,j}$, $\overline{b}^{g,\varepsilon}_i$, $\partial_k \overline{A}^{g,\varepsilon}_{i,j}$ et $\overline{c}^{g,\varepsilon}$ appartiennent à $\mathcal{H}^{\delta,\delta/2}([0,T]\times\mathbb{R}^3)$, il découle que, pour $|f| \leq M$ et $|\xi| \leq M_1$, les fonctions α_i, α, $\frac{\partial \alpha_i}{\partial \xi_j}$ et $\frac{\partial \alpha_i}{\partial v_j}$ sont höldériennes en t, v, f et ξ, d'ordre $\delta/2$, δ, δ et δ respectivement. On a également la relation

$$\forall \xi \in \mathbb{R}^3, \qquad \varepsilon\,|\xi|^2 \leq \sum_{i,j} (\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\delta_{i,j})\,\xi_i\xi_j \leq (\,3\,K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon\,)\,|\xi|^2, \tag{A.2.4}$$

car $\|a^\varepsilon_{i,j}\|_{\mathcal{C}^4_b} \leq K_\varepsilon$. On applique le Théorème 5.8.1 de [1] pour obtenir l'existence d'une solution f^ε de (A.2.2)-(A.2.3) appartenant à l'espace de Hölder $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$. De plus, il existe une constante Λ ne dépendant que de f_{in}, δ, ε et C_L telle que (A.2.1) soit vérifié.

Montrons maintenant que

$$0 \leq f^\varepsilon(t,v) \leq 1. \tag{A.2.5}$$

Soit $\mathcal{L}_1$ l'opérateur linéaire défini par

$$\mathcal{L}_1 u = \partial_t u - \sum_{i,j} \Big(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\,\delta_{i,j}\Big)\,\partial^2_{i,j} u - \sum_i \Big[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2f^\varepsilon)\Big]\partial_i u + \overline{c}^{g,\varepsilon}\,\theta(f^\varepsilon)\,u.$$

Soit $R>0$. Comme $\mathcal{L}_1\Big(\big(|v|^2+Ct\big)e^{\lambda t}\Big) \geq 0$, dès que

$$C \geq 6K_\varepsilon\|f_{in}\|_{L^1} + 6\varepsilon + 12\,(1+\Lambda)^2 K_\varepsilon^2 \|f_{in}\|^2_{L^1} \quad \text{et} \quad \lambda \geq 1 + K_\varepsilon \|f_{in}\|_{L^1},$$

on a

$$\mathcal{L}_1\Big(-\frac{\Lambda}{R^2}\,\big(|v|^2+Ct\big)e^{\lambda t}\Big) = -\frac{\Lambda}{R^2}\,\mathcal{L}_1\Big(\big(|v|^2+Ct\big)e^{\lambda t}\Big) \leq 0 = \mathcal{L}_1(f^\varepsilon);$$
$$-\frac{\Lambda}{R^2}|v|^2 \leq 0 \leq f_{in}(v);$$
$$-\frac{\Lambda}{R^2}\big(|v|^2+Ct\big)e^{\lambda t} \leq -\Lambda \leq f^\varepsilon(t,v), \quad \text{si } |v| = R.$$

On déduit du principe de comparaison [1, Theorem 1.2.1]) que

$$f^\varepsilon(t,v) \geq -\frac{\Lambda}{R^2}\,\big(|v|^2+Ct\big)e^{\lambda t}, \quad \forall (t,v) \in [0,T]\times B_R.$$

On fait tendre R vers l'infini et on obtient

$$f^\varepsilon(t,v) \geq 0, \qquad \forall (t,v) \in [0,T]\times\mathbb{R}^3.$$

Soit $\mathcal{L}_2$ l'opérateur quasi-linéaire défini par

$$\mathcal{L}_2 u = \partial_t u - \sum_{i,j} \left(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\,\delta_{i,j}\right) \partial_{i,j}^2 u - \sum_i \left[\overline{B}_i^{g,\varepsilon} - \overline{b}_i^{g,\varepsilon}(1-2f^\varepsilon)\right]\partial_i u + \overline{c}^{g,\varepsilon} f^\varepsilon\,\theta(u).$$

Soit $R > 0$. Comme

$$\mathcal{L}_2 \left(1 + \frac{\Lambda}{R^2}\left(|v|^2 + Ct\right)e^t\right) \geq 0 = \mathcal{L}_2(f^\varepsilon),$$

pour

$$C \geq 6K_\varepsilon \|f_{in}\|_{L^1} + 6\varepsilon + 12\,(1+\Lambda)^2 K_\varepsilon^2 \|f_{in}\|_{L^1}^2,$$

et comme

$$1 + \frac{\Lambda}{R^2}|v|^2 \geq 1 \geq f_{in}(v);$$
$$1 + \frac{\Lambda}{R^2}\left(|v|^2 + Ct\right)e^t \geq \Lambda \geq f^\varepsilon(t,v), \qquad \text{si } |v| = R,$$

on déduit du principe de comparaison pour les équations quasi-linéaires (cf. [2, Theorem 9.1]) que

$$f^\varepsilon(t,v) \leq 1 + \frac{\Lambda}{R^2}\left(|v|^2 + Ct\right)e^t, \qquad \forall (t,v) \in [0,T]\times B_R.$$

On fait tendre R vers l'infini et on obtient, pour tout (t,v) de $[0,T]\times\mathbb{R}^3$, $f^\varepsilon(t,v) \leq 1$.

On déduit de (A.2.5) que la fonction f^ε est solution du problème (A.1.1)-(A.1.2).

Montrons maintenant l'unicité. Soient h_1 et h_2 deux solutions du problème (A.1.1)-(A.1.2) appartenant à $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$. Alors, $u = h_1 - h_2$ vérifie

$$\partial_t u - \sum_{i,j}\left(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\delta_{i,j}\right)\partial_{i,j}^2 u - \sum_j \left[\overline{B}_j^{g,\varepsilon} - \overline{b}_j^{g,\varepsilon}(1-h_1-h_2)\right]\partial_j u$$
$$+ \left[\overline{c}^{g,\varepsilon}(1-h_1-h_2) - \sum_j \overline{b}_j^{g,\varepsilon}(\partial_j h_1 + \partial_j h_2)\right]u = 0. \quad \text{(A.2.6)}$$

On déduit du Théorème 1.2.5 de [1] que $h_1 = h_2$. □

A.3 Appartenance à l'ensemble $\mathcal{C}$

On montre ici que, pour un choix convenable de $\beta_1' \geq \beta_1$, D, E, F et C_L les solutions de (A.1.1)-(A.1.2) appartiennent à l'ensemble $\mathcal{C}$. Voici tout d'abord une conséquence du Théorème 9.1 de [2] qui servira pour montrer que la solution f^ε vérifie la majoration par une distribution de Fermi-Dirac.

Lemme A.3.1 *Soit $\mathcal{P}$ l'opérateur quasi-linéaire défini par*

$$\mathcal{P}u = \partial_t u - \sum_{i,j} \alpha_{i,j}(t,v)\,\partial^2_{i,j} u - \sum_i \Big[\beta_i(t,v) - \chi_i(t,v)(1-2u)\Big]\partial_i u + \mu(t,v)(1-u)\,u,$$

où les coefficients $\alpha_{i,j}$, β_i, χ_i et μ sont des fonctions continues sur $[0,T]\times\mathbb{R}^3$. Soient g et h deux fonctions de $\mathcal{C}^{1,2}\big([0,T]\times\mathbb{R}^3\big)\cap L^\infty((0,T)\times\mathbb{R}^3)$ telles que,

$$\mathcal{P}g \leq \mathcal{P}h \text{ dans } (0,T]\times\mathbb{R}^3, \qquad g \leq h \text{ sur } \{0\}\times\mathbb{R}^3$$

et telles qu'il existe une fonction $\Phi \in \mathcal{C}\big(\mathbb{R}^3\big)$ vérifiant

$$\lim_{|v|\to+\infty} \Phi(v) = 0 \quad \text{ et } \quad \forall (t,v)\in[0,T]\times\mathbb{R}^3, \quad (g-h)(t,v) \leq \Phi(v).$$

On suppose de plus que les coefficients $\alpha_{i,j}$ vérifient

$$\forall\, \xi\in\mathbb{R}^3, \quad \forall\,(t,v)\in[0,T]\times\mathbb{R}^3, \qquad \sum_{i,j}\alpha_{i,j}(t,v)\xi_i\,\xi_j \geq 0,$$

et que l'on a, soit

$$\sup_{(t,v)\in[0,T]\times\mathbb{R}^3}\left| 2\sum_i \chi_i(t,v)\,\partial_i g(t,v) + \mu(t,v)\,(g+h-1)(t,v)\right| < \infty, \tag{A.3.1}$$

soit

$$\sup_{(t,v)\in[0,T]\times\mathbb{R}^3}\left| 2\sum_i \chi_i(t,v)\,\partial_i h(t,v) + \mu(t,v)\,(g+h-1)(t,v)\right| < \infty. \tag{A.3.2}$$

Alors, on a $g\leq h$ dans $[0,T]\times\mathbb{R}^3$.

Preuve. On adapte ici la démonstration de [2]. Supposons, par exemple, que l'hypothèse (A.3.1) est vérifiée. Posons $w=(g-h)\,e^{\lambda t}$, où le réel λ vérifie

$$\lambda < -\sup_{(t,v)\in[0,T]\times\mathbb{R}^3}\left| 2\sum_i \chi_i(t,v)\,\partial_i g(t,v) + \mu(t,v)\,(g+h-1)(t,v)\right|.$$

On a $w\leq 0$ sur $\{0\}\times\mathbb{R}^3$. Supposons qu'il existe un point (t_0,v_0) de $(0,T]\times\mathbb{R}^3$ tel que $w(t_0,v_0)>0$. Pour tout (t,v) de $[0,T]\times\mathbb{R}^3$, on a $w(t,v)\leq\Phi(v)\,e^{\lambda T}$. Il existe donc $R>0$ tel que, pour $|v|\geq R$, on ait $w(t,v)\leq w(t_0,v_0)/2$. La fonction w atteint alors son maximum sur $(0,T]\times B_R$. Il existe un point (t_1,v_1) de $(0,T]\times B_R$ tel que

$$\begin{cases} w(t_1,v_1) > 0, \\ \nabla g(t_1,v_1) = \nabla h(t_1,v_1), \\ \nabla^2(g-h)(t_1,v_1) \leq 0, \\ \partial_t(g-h)(t_1,v_1) \geq -\lambda(g-h)(t_1,v_1), \end{cases}$$

où $\nabla^2(g-h)$ désigne la matrice hessienne de $g-h$. On déduit alors que

$$\begin{aligned}0 \le \mathcal{P}h(t_1,v_1) - \mathcal{P}g(t_1,v_1) &= -\partial_t(g-h)(t_1,v_1) + \sum_{i,j} \alpha_{i,j}(t_1,v_1)\,\partial^2_{i,j}(g-h)(t_1,v_1)\\ &+ \Big[2\sum_i \chi_i(t_1,v_1)\partial_i g(t_1,v_1) + \mu(t_1,v_1)(g+h-1)(t_1,v_1)\Big](g-h)(t_1,v_1).\end{aligned}$$

D'où, la contradiction

$$\begin{aligned}0 &\le \mathcal{P}h(t_1,v_1) - \mathcal{P}g(t_1,v_1)\\ &\le \Big[\lambda + \sup_{(t,v)\in[0,T]\times\mathbb{R}^3}\Big|2\sum_i \chi_i(t,v)\partial_i g(t,v) + \mu(t,v)(g+h-1)(t,v)\Big|\Big](g-h)(t_1,v_1)\\ &< 0.\end{aligned}$$

Ce qui implique donc que $g \le h$ dans $[0,T]\times\mathbb{R}^3$. On aurait eu la même conclusion si on avait supposé que (A.3.2) était vérifiée car $\nabla g(t_1,v_1) = \nabla h(t_1,v_1)$. □

Lemme A.3.2 *Pour un choix convenable de $\beta_1' \ge \beta_1$, D, E et F, les solutions f^ε de (A.1.1)-(A.1.2) vérifient, pour tout (t,v) de $[0,T]\times\mathbb{R}^3$,*

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f^\varepsilon(t,v) \le \frac{\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}{1+\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}.$$

Preuve. Posons

$$\varphi_{inf}(t,v) = \alpha_1\, e^{-\beta_1'|v|^2}\, e^{-Dt} \qquad \text{et} \qquad \varphi_{sup}(t,v) = \frac{\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}{1+\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}.$$

On a

$$\begin{aligned}\mathcal{L}_1\varphi_{inf} = \Bigg\{ &-4\,\beta_1'^2 \sum_{i,j}\big(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\delta_{i,j}\big)\, v_i v_j + 2\,\beta_1' \sum_i \Big[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2f^\varepsilon)\Big] v_i\\ &- D + 2\,\beta_1'\left(\sum_i \overline{A}^{g,\varepsilon}_{i,i} + 3\varepsilon\right) + \overline{c}^{g,\varepsilon}(1-f^\varepsilon)\Bigg\}\varphi_{inf}.\end{aligned}$$

Or, comme $\|\overline{B}^{g,\varepsilon}_i\|_{L^\infty}$ et $\|\overline{b}^{g,\varepsilon}_i\|_{L^\infty}$ sont bornées par $K_\varepsilon\|f_{in}\|_{L^1}$, on obtient

$$\begin{aligned}\left|2\sum_i\Big[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2f^\varepsilon)\Big] v_i\right| &\le |v|^2 + \sum_i\Big|\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i(1-2f^\varepsilon)\Big|^2\\ &\le |v|^2 + 12\,K_\varepsilon^2\|f_{in}\|^2_{L^1},\end{aligned}$$

ce qui, avec l'ellipticité (A.2.4), implique que

$$\mathcal{L}_1\varphi_{inf} \leq \Bigg\{ - \Big[4\,\beta_1'\varepsilon - 1\Big]\,\beta_1'\,|v|^2 + 12\beta_1' K_\varepsilon^2 \|f_{in}\|_{L^1}^2 + 6\beta_1'\big(K_\varepsilon\|f_{in}\|_{L^1} + \varepsilon\big) + K_\varepsilon\|f_{in}\|_{L^1} - D\Bigg\}\varphi_{inf}.$$

Par conséquent, pour

$$\beta_1' \geq \frac{1}{4\varepsilon} \qquad \text{et} \qquad D \geq 12\beta_1' K_\varepsilon^2 \|f_{in}\|_{L^1}^2 + 6\beta_1'(K_\varepsilon\|f_{in}\|_{L^1} + \varepsilon) + K_\varepsilon\|f_{in}\|_{L^1},$$

on a $\mathcal{L}_1\varphi_{inf} \leq 0$. Soit $\eta > 0$. Posons

$$w(t,v) = \varphi_{inf}(t,v) - \eta e^{\lambda t} - f^\varepsilon(t,v),$$

avec $\lambda \geq K_\varepsilon\,\|f_{in}\|_{L^1}$. On a alors $\mathcal{L}_1 w \leq 0$. Comme $w(0,v) \leq 0$ par (A.1.3) et comme il existe une constante R_η tel que

$$w(t,v) \leq 0 \quad \text{sur} \quad [0,T] \times \big(\mathbb{R}^3 \backslash B_{R_\eta}\big),$$

on déduit du principe de comparaison (cf. [1, Theorem 1.2.1]) que, pour tout (t,v) de $[0,T] \times \mathbb{R}^3$, $w(t,v) \leq 0$, et donc

$$\varphi_{inf}(t,v) - \eta e^{\lambda t} \leq f^\varepsilon(t,v).$$

En faisant tendre η vers 0, on conclut que

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \leq f^\varepsilon(t,v), \qquad \forall (t,v) \in [0,T] \times \mathbb{R}^3.$$

Avant de montrer la majoration de f^ε par φ_{sup}, nous allons montrer qu'il existe des constantes $Q > 0$ et $S > 0$ telles que

$$f^\varepsilon(t,v) \leq \alpha_2\, e^{\frac{-\beta_2|v|^2}{1+St}}\, e^{Qt} = \theta_{sup}(t,v),$$

ce qui nous permettra d'appliquer le Lemme A.3.1. On a

$$\mathcal{L}_1\theta_{sup} = \Bigg\{ \frac{\beta_2}{(1+St)^2}\Big[S|v|^2 - 4\beta_2 \sum_{i,j} \big(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\delta_{i,j}\big) v_i v_j\Big] + \frac{2\beta_2}{1+St}\Bigg(\sum_i \overline{A}_{i,i}^{g,\varepsilon} + 3\varepsilon\Bigg) + Q + \frac{2\beta_2}{1+St}\sum_i \Big[\overline{B}_i^{g,\varepsilon} - \overline{b}_i^{g,\varepsilon}(1-2f^\varepsilon)\Big]\, v_i + \overline{c}^{g,\varepsilon}(1-f^\varepsilon)\Bigg\}\theta_{sup}.$$

On déduit alors, comme précédemment,

$$\mathcal{L}_1\theta_{sup} \geq \Bigg\{ \frac{\beta_2|v|^2}{(1+St)^2}\Big[S - 4\,\beta_2(\,3\,K_\varepsilon\|f_{in}\|_{L^1} + \varepsilon\,) - \beta_2\Big] + Q - 12K_\varepsilon^2\|f_{in}\|_{L^1}^2 - K_\varepsilon\|f_{in}\|_{L^1}\Bigg\}\theta_{sup}.$$

Par conséquent, pour

$$Q \geq 12K_\varepsilon^2 \|f_{in}\|_{L^1}^2 + K_\varepsilon \|f_{in}\|_{L^1} \quad \text{et} \quad S \geq 4\,\beta_2(\,3\,K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon\,) + \beta_2,$$

on a $\mathcal{L}_1 \theta_{sup} \geq 0$. Soit $R > 0$. Posons

$$w(t,v) = f^\varepsilon(t,v) - \frac{1}{R^2}\big(|v|^2 + Ct\big)e^{\lambda t} - \theta_{sup}(t,v).$$

Pour $\lambda \geq 1 + K_\varepsilon \|f_{in}\|_{L^1}$ et $C \geq 6K_\varepsilon \|f_{in}\|_{L^1} + 6\varepsilon + 12K_\varepsilon^2 \|f_{in}\|_{L^1}^2$, on obtient

$$\mathcal{L}_1\big(\,\big(|v|^2 + Ct\big)e^{\lambda t}\,\big) \geq 0.$$

Ainsi, on a

$$\begin{array}{rcll} \mathcal{L}_1 w & \leq & 0 & \text{sur} \quad [0,T] \times B_R \\ w & \leq & 0 & \text{sur} \quad \partial([0,T] \times B_R). \end{array}$$

Du principe de comparaison (cf. [1, Theorem 1.2.1]), on déduit alors que, pour tout (t,v) de $[0,T] \times B_R$,

$$f^\varepsilon(t,v) \leq \theta_{sup}(t,v) + \frac{1}{R^2}\big(|v|^2 + Ct\big)e^{\lambda t}.$$

On fait tendre R vers l'infini et on déduit

$$f^\varepsilon(t,v) \leq \alpha_2\, e^{\frac{-\beta_2 |v|^2}{1+St}}\, e^{Qt} = \theta_{sup}(t,v).$$

Soit $\mathcal{M}$ l'opérateur quasi-linéaire défini par

$$\mathcal{M}u = \partial_t u - \sum_{i,j} \left(\overline{A}_{i,j}^{g,\varepsilon} + \varepsilon\, \delta_{i,j}\right) \partial_{i,j}^2 u - \sum_i \left[\overline{B}_i^{g,\varepsilon} - \overline{b}_i^{g,\varepsilon}(1-2u)\right]\partial_i u + \overline{c}^{g,\varepsilon}(1-u)\,u.$$

On a

$$\begin{aligned} \mathcal{M}\varphi_{sup} = \Bigg\{ \frac{\beta_2 |v|^2}{(1+Ft)^2} \Bigg[F + 4\beta_2\, \frac{\alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}} - 1}{1 + \alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}} \sum_{i,j} \big(\overline{A}_{i,j}^{\varepsilon} + \varepsilon \delta_{i,j}\big) \frac{v_i v_j}{|v|^2} \Bigg] \\ + \frac{2\beta_2}{1+Ft} \sum_i \left[\overline{B}_i^{\varepsilon} - \overline{b}_i^{\varepsilon}(1 - 2\varphi_{sup})\right] v_i \\ + \left(\sum_i \overline{A}_{i,i}^{g,\varepsilon} + 3\varepsilon\right) \frac{2\beta_2}{1+Ft} + E + \overline{c}^{\varepsilon} \Bigg\} \frac{\varphi_{sup}}{1 + \alpha_2\, e^{Et}\, e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}. \end{aligned}$$

Or,

$$\Big(\sum_i \overline{A}^{g,\varepsilon}_{i,i} + 3\varepsilon\Big)\frac{2\beta_2}{1+Ft} \geq 0$$

$$\frac{4\,\beta_2\,\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}{1+\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}\sum_{i,j}\big(\overline{A}^{\varepsilon}_{i,j}+\varepsilon\delta_{i,j}\big)\frac{v_iv_j}{|v|^2} \geq 0$$

$$\left|\frac{4\beta_2}{1+\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}\sum_{i,j}\big(\overline{A}^{\varepsilon}_{i,j}+\varepsilon\delta_{i,j}\big)\frac{v_iv_j}{|v|^2}\right| \leq 4\,\beta_2\,(3\,K_\varepsilon\,\|f_{in}\|_{L^1}+\varepsilon)$$

$$\left|\frac{2\beta_2}{1+Ft}\sum_i\Big[\overline{B}^{\varepsilon}_i-\overline{b}^{\varepsilon}_i(1-2\varphi_{sup})\Big]\,v_i\right| \leq 12\,K_\varepsilon^2\,\|f_{in}\|^2_{L^1}+\frac{\beta_2^2\,|v|^2}{(1+Ft)^2}.$$

Donc, on déduit

$$\mathcal{M}\varphi_{sup} \geq \Bigg\{\frac{\beta_2\,|v|^2\big(F-12\beta_2K_\varepsilon\|f_{in}\|_{L^1}-4\,\beta_2\,\varepsilon-\beta_2\big)}{(1+Ft)^2} + E-12K_\varepsilon^2\,\|f_{in}\|^2_{L^1}-K_\varepsilon\,\|f_{in}\|_{L^1}\Bigg\}\frac{\varphi_{sup}}{1+\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}.$$

Par conséquent, pour

$$E \geq 12K_\varepsilon^2\,\|f_{in}\|_{L^1}+K_\varepsilon\,\|f_{in}\|^2_{L^1} \qquad \text{et} \qquad F \geq 12\,\beta_2K_\varepsilon\|f_{in}\|_{L^1}+4\,\beta_2\,\varepsilon+\beta_2,$$

on obtient $\mathcal{M}\varphi_{sup} \geq 0$. Or, pour tout v de $\mathbb{R}^3$, on a

$$\varphi_{sup}(0,v) = \frac{\alpha_2\,e^{-\beta_2|v|^2}}{1+\alpha_2 e^{-\beta_2|v|^2}} \geq f_{in}(v).$$

On applique alors le lemme A.3.1 car on a la majoration par θ_{sup} et on obtient, pour tout (t,v) de $[0,T]\times\mathbb{R}^3$,

$$f^\varepsilon(t,v) \leq \frac{\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}{1+\alpha_2\,e^{Et}\,e^{-\frac{\beta_2\,|v|^2}{1+Ft}}}.$$

□

Pour montrer que f^ε appartient à $\mathcal{C}$, il nous reste à montrer les deux conditions de Lipschitz ainsi que les conditions

$$f^\varepsilon \in \mathcal{C}\big([0,T];L^1\big(\mathbb{R}^3\big)\big), \tag{A.3.3}$$

$$\int_{\mathbb{R}^3} f^\varepsilon(t,v)\,dv = \int_{\mathbb{R}^3} f_{in}(v)\,dv, \qquad t\in[0,T]. \tag{A.3.4}$$

En ce qui concerne (A.3.3), c'est une conséquence immédiate du Lemme A.3.2 et de la continuité de f^ε. Quant à l'égalité (A.3.4), elle découle du Lemme A.3.2 et d'arguments de troncature. Finalement, une preuve détaillée des conditions de Lipschitz a déjà été donnée dans le Chapitre 2 (cf. Lemme 2.4.7).

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Chapitre 3

Equilibrium states for the Landau-Fermi-Dirac equation

Article en collaboration avec Mohammed Lemou faisant partie des actes du colloque "Non-local elliptic and parabolic problems" parus dans les Banach Center Publications, Volume 66, Pages 29-37, Année 2004.

Abstract

A kinetic collision operator of Landau type for Fermi-Dirac particles is considered. Equilibrium states are rigorously determined under minimal assumptions on the distribution function of the particles. The particular structure of the considered operator (strong non-linearity and degeneracy) requires a special investigation compared to the classical Boltzmann or Landau operator.

3.1 Introduction

The Landau or Landau-Fokker-Planck equation is a kinetic collision model used to describe the evolution of charged particles in a plasma [2, 3, 4, 11]. When quantum effects such as the Pauli exclusion principle come into play, this collision operator has to be modified and leads to the so-called Landau-Fermi-Dirac (LFD) operator [4, 6, 11]. Besides, a Landau equation with Fermi statistics also arises in the modelling of stellar systems [5, 9]. In this paper, we consider the LFD equation in the spatially homogeneous case. It reads:

$$\partial_t f(t,v) = Q_L(f)(t,v), \qquad t \in \mathbb{R}_+, v \in \mathbb{R}^3, \tag{3.1.1}$$

where

$$Q_L(f)(t,v) = \nabla \cdot \int \Psi(v-v_*)\, \Pi(v-v_*) \Big\{ f_*(1-f_*)\nabla f - f(1-f)\nabla f_* \Big\}\, dv_*, \tag{3.1.2}$$

with $f = f(t,v)$, $f_* = f(t,v_*)$, $\Pi(z)$ denotes the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}, \qquad 1 \le i,j \le 3,$$

and Ψ is a function such as $\Psi(z) = |z|^{2+\gamma}$, $-3 \le \gamma \le 1$. Here as in the rest of this paper, ∇ denotes the gradient with respect to the v variable. The choice $\Psi(z) = |z|^{2+\gamma}$ corresponds to inverse power law potentials. According to the value of γ, we distinguish the Coulomb potential ($\gamma = -3$), soft potentials ($-3 < \gamma < 0$), the Maxwellian potential ($\gamma = 0$) and hard potentials ($0 < \gamma \le 1$). We recall here that the Coulomb potential is nevertheless the only one to have a physical relevance.

Equilibrium states and trend to equilibrium for the classical Boltzmann and Landau equations have been considered in several papers, see [3, 7, 14, 15] for the Boltzmann equation and [8, 16, 17] for the Landau equation, and the references therein. For the Boltzmann-Fermi-Dirac (BFD) equation, Lu [12] has shown the existence of two classes of equilibria, which are the class of Fermi-Dirac distributions and the class of characteristic functions of the Euclidean balls. Large time behaviour for the BFD equation has been studied in [13]. To our knowledge, there are few works on the Landau-Fermi-Dirac equation ([6, 10, 1]). In particular, the determination of its equilibrium states has not been yet considered at a rigorous level. We point out that the Pauli exclusion principle implies that a solution to both LFD and BFD equations must satisfy $0 \le f \le 1$ as soon as this is satisfied by the initial data. Similarly to the BFD equation, there should be two classes of equilibria for the LFD equation, namely the class of Fermi-Dirac distributions and a class of degenerate equilibria. Our purpose in this present work is to clarify this claim. In particular, we rigorously determine the expressions of the equilibrium states (i.e. the solutions to $Q_L(f) = 0$) under minimal and 'natural' assumption on the distribution function f. The strong non-linearity in (3.1.2) (term $f(1-f)$) and its degeneracy for $f \sim 1$ give rise to additional difficulties compared to the classical case and a special treatment is required.

We now describe the contents of the paper. We set notations and state our main result in the next section. The proof is given in Section 3.3.

3.2 Main results

The usual *a priori* estimates are available for (3.1.1)-(3.1.2). Indeed, one can formally check that solutions preserve mass and energy, namely

$$\forall t \geq 0, \qquad \int f(t,v)\,dv = \int f_{in}\,dv \qquad \text{and} \qquad \int f(t,v)\,|v|^2\,dv = \int f_{in}\,|v|^2\,dv.$$

Moreover, considering the entropy for Fermi-Dirac particles defined by

$$S(f) = -\int \Big[f \ln f + (1-f)\ln(1-f) \Big]\,dv \geq 0,$$

one can see, still formally, that $t \longmapsto S(f)(t)$ is a non-decreasing function. More generally, the dissipation term reads

$$\int Q_L(f)\Big[\ln(1-f) - \ln f\Big]\,dv = \frac{1}{2}\iint \Pi(v-v_*)|v-v_*|^{\gamma+2} \Big(f_*(1-f_*)\nabla f - f(1-f)\nabla f_*\Big)\left(\frac{\nabla f}{f(1-f)} - \frac{\nabla f_*}{f_*(1-f_*)}\right)\,dv_*\,dv.$$

The conservation of mass and energy and the fact that the entropy is a non-decreasing function have been rigorously proved in [1] for solutions to (3.1.1)-(3.1.2) with $0 \leq \gamma \leq 1$.

Equilibrium states are usually defined thanks to the cancellation of the dissipation term. The problem here is to give a sense to this expression. Noting that

$$2\,\nabla\left[\,\text{Arctan}\,\sqrt{\frac{f}{1-f}}\,\right] = \frac{\nabla f}{\sqrt{f(1-f)}}$$

and that Π is a projector and thus satisfies $\Pi = \Pi^2$, we infer that

$$\int Q_L(f)\Big[\ln(1-f) - \ln f\Big]\,dv = 2\iint \Big|\Pi(v-v_*)|v-v_*|^{(2+\gamma)/2}\Big[g_*\nabla\big(p(f)\big) - g\nabla_*\big(p(f_*)\big)\Big]\Big|^2\,dv_*\,dv,$$

where $g = \sqrt{f(1-f)}$, $p(f) =$ Arctan $\big(\sqrt{f/(1-f)}\big)$ and ∇_* denotes the gradient with respect to the v_* variable.

If f is a measurable function satisfying $0 \leq f \leq 1$ a.e. then $p(f)$ belongs to $L^\infty(\mathbb{R}^3)$. Consequently, $\nabla p(f) \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. We may now define what we mean by equilibrium states. We consider

$$\Omega = \Big\{(v,v_*) \in \big(\mathbb{R}^3\big)^2\ ;\ v \neq v_*\Big\}.$$

Definition 3.2.1 *A function $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is said to be an equilibrium state for the LFD equation if it satisfies $0 \le f \le 1$ a.e. and*

$$\Pi(v - v_*)\, |v - v_*|^{(2+\gamma)/2} \Big[g_*\, \nabla\big(p(f)\big) - g\, \nabla_*\big(p(f_*)\big)\Big] = 0, \qquad in\ \mathcal{D}'\big(\Omega, \mathbb{R}^3\big). \quad (3.2.1)$$

Formally, if f is a smooth function that satisfies $0 \le f \le 1$ a.e. and (3.2.1), then

$$f(v) = \frac{a e^{-b|v-V_0|^2}}{1 + a e^{-b|v-V_0|^2}},$$

with $a, b > 0$ and $V_0 \in \mathbb{R}^3$. Our aim is to give a rigorous proof for this statement, under 'minimal' assumptions for f.

Remark 3.2.2 *Any function $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that $0 \le f \le 1$ a.e. and $f(1-f) = 0$ a.e. satisfies (3.2.1), that is any characteristic function of a measurable set with a finite measure is a solution to (3.2.1). We thus recover a class of degenerate equilibria as for the BFD equation (see [12]). However, this new class strictly includes the one concerning the BFD equation.*

Owing to the previous remark, we restrict ourselves to the functions that satisfy (3.2.1) and

$$\text{meas}\big(\{v \in \mathbb{R}^3\,;\ 0 < f(v) < 1\}\big) \neq 0. \quad (3.2.2)$$

Our main result is the following.

Theorem 3.2.3 *The equilibrium states of the LFD equation satisfying (3.2.2) are the Fermi-Dirac distributions, that is the functions of the following form:*

$$f(v) = \frac{a e^{-b|v-V_0|^2}}{1 + a e^{-b|v-V_0|^2}},$$

with $V_0 \in \mathbb{R}^3$ and $a, b > 0$.

3.3 Proof of Theorem 3.2.3

Let $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying (3.2.1), (3.2.2) and $0 \le f \le 1$ a.e. on $\mathbb{R}^3$. We set

$$T = g_*\, \nabla(p(f)) - g\, \nabla_*(p(f_*)).$$

Then, (3.2.1) implies that

$$\Pi(v - v_*)\, T = 0 \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^3). \quad (3.3.1)$$

Lemma 3.3.1 *If (3.3.1) holds, there exists a real-valued distribution $\Lambda_{v,v_*} \in \mathcal{D}'(\Omega, \mathbb{R})$ such that*

$$T = (v - v_*)\, \Lambda_{v,v_*}, \qquad in\ \mathcal{D}'(\Omega, \mathbb{R}^3). \quad (3.3.2)$$

Proof. The proof of this lemma is similar to that of the classical case [17]. Let $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$. Since $\Pi(z)$ is the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\varphi(v, v_*) = \lambda(v, v_*)(v - v_*) + \zeta(v, v_*),$$

with

$$\begin{aligned} \zeta(v, v_*) &= \Pi(v - v_*)\, \zeta(v, v_*) = \Pi(v - v_*)\, \varphi(v, v_*), \\ \lambda(v, v_*) &= \frac{\varphi(v, v_*) \cdot (v - v_*)}{|v - v_*|^2}. \end{aligned}$$

Then,

$$\begin{aligned} \left\langle T, \varphi(v, v_*) \right\rangle &= \left\langle (v - v_*) \cdot T \,,\, \lambda(v, v_*) \right\rangle + \left\langle T \,,\, \Pi(v - v_*)\, \zeta(v, v_*) \right\rangle \\ &= \left\langle (v - v_*) \cdot T \,,\, \frac{\varphi(v, v_*) \cdot (v - v_*)}{|v - v_*|^2} \right\rangle + \left\langle \Pi(v - v_*)\, T \,,\, \zeta(v, v_*) \right\rangle, \end{aligned}$$

where $\langle \ , \ \rangle$ denotes the dual product. Owing to (3.3.1), equation (3.3.2) holds for

$$\Lambda_{v,v_*} = \frac{(v - v_*) \cdot T}{|v - v_*|^2}. \qquad \square$$

Lemma 3.3.2 *Let $\mathcal{P}$ be a measurable set with a positive measure. Then, there exist distinct points $u_i \in \mathbb{R}^3$, $i = 1, 2, 3$ such that, for $i = 1, 2, 3$, we have*

$$\forall r > 0, \quad meas\big(B(u_i, r) \cap \mathcal{P}\big) > 0, \tag{3.3.3}$$

where $B(u_i, r)$ denotes the ball with centre u_i and radius r of $\mathbb{R}^3$.

Moreover, there exist $r_i > 0$, $i = 1, 2, 3$ such that

$$B_i \cap B_j = \emptyset, \quad if \quad i \neq j, \tag{3.3.4}$$

where $B_i := B(u_i, r_i)$, $i = 1, 2, 3$.

Proof.
Step 1. We first prove that there exists $u_1 \in \mathbb{R}^3$ that satisfies (3.3.3). Suppose, contrary to our claim, that for every $w \in \mathbb{R}^3$ there exists $r(w) > 0$ such that $\text{meas}\big(B(w, r(w)) \cap \mathcal{P}\big) = 0$. Then, for $n \in \mathbb{N}$,

$$B(0, n) \subset \bigcup_{w \in B(0,n)} B(w, r(w)).$$

Since $B(0, n)$ is relatively compact in $\mathbb{R}^3$, there exist some w_i, $i = 1, \ldots, N$, such that

$$B(0, n) \subset \bigcup_{i=1}^{N} B(w_i, r(w_i)).$$

Hence,

$$\text{meas}\big(B(0,n)\cap\mathcal{P}\big) \le \sum_{i=1}^{N} \text{meas}\big(B(w_i, r(w_i))\cap\mathcal{P}\big) = 0$$

and $\text{meas}\big(\mathcal{P}\big) = \lim_{n\to\infty}\text{meas}\big(B(0,n)\cap\mathcal{P}\big) = 0$, which contradicts our assumption on $\mathcal{P}$. Consequently, there exists $u_1 \in \mathbb{R}^3$ that satisfies (3.3.3).

Step 2. The function τ defined by $\tau(r) = \text{meas}\big(B(u_1, r)\cap\mathcal{P}\big)$ is continuous and satisfies $\tau(0) = 0$ and $\lim_{r\to+\infty}\tau(r) = \text{meas}\big(\mathcal{P}\big)$. Therefore, there exists $r_1 > 0$ such that

$$\text{meas}\big(B(u_1, 2\,r_1)\cap\mathcal{P}\big) \le \frac{\text{meas}\big(\mathcal{P}\big)}{4}. \tag{3.3.5}$$

We set $\mathcal{P}_1 := \mathcal{P}\backslash B(u_1, 2r_1)$. From (3.3.5) follows that $\text{meas}(\mathcal{P}_1) \ge 3\,\text{meas}(\mathcal{P})/4 > 0$. Similarly to the first step, we infer that there exists $u_2 \in \mathbb{R}^3\backslash B(u_1, 2r_1)$ such that

$$\forall r > 0, \quad \text{meas}\big(B(u_2, r)\cap\mathcal{P}_1\big) > 0.$$

Since $\mathcal{P}_1 \subset \mathcal{P}$, u_2 also satisfies (3.3.3). As previously, there exists $\overline{r}_2 > 0$ such that

$$\text{meas}\big(B(u_2, 2\,\overline{r}_2)\cap\mathcal{P}\big) \le \frac{\text{meas}\big(\mathcal{P}\big)}{4}.$$

We choose $r_2 := \min(\overline{r}_2, d(u_2, B(u_1, r_1)))$, where $d(u_2, B(u_1, r_1))$ denotes the distance between u_2 and $B(u_1, r_1)$.

We now set $\mathcal{P}_2 := \mathcal{P}\backslash\big(B(u_1, 2r_1)\cup B(u_2, 2r_2)\big)$. Then, $\text{meas}(\mathcal{P}_2) \ge \text{meas}(\mathcal{P})/2 > 0$. Similarly to the first step, it implies that there exists $u_3 \in \mathbb{R}^3\backslash(B(u_1, 2r_1)\cup B(u_2, 2r_2))$ such that

$$\forall r > 0, \quad \text{meas}\big(B(u_3, r)\cap\mathcal{P}_2\big) > 0.$$

Since $\mathcal{P}_2 \subset \mathcal{P}$, u_3 satisfies (3.3.3). We set $r_3 := \min(d(u_3, B(u_1, r_1)), d(u_3, B(u_2, r_2)))$. □

Proposition 3.3.3 *Let $f \in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ satisfying $0 \le f \le 1$ a.e., (3.2.2) and (3.3.1). Then $f \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$.*

Proof. We consider

$$U = \left\{(v_1, v_2, v_3) \in (\mathbb{R}^3)^3\ ;\ v_1 \ne v_2,\ v_1 \ne v_3,\ v_2 \ne v_3\right\},$$

and for $(v_1, v_2, v_3) \in U$, we set $f_i = f(v_i)$, $g_i = \sqrt{f_i(1-f_i)}$ and $\Lambda_{i,j} = \Lambda_{v_i,v_j}$, $i,j = 1,2,3$. We deduce from Lemma 3.3.1 that

$$\begin{aligned}
g_2\,g_3\,\nabla p(f_1) - g_1\,g_3\,\nabla p(f_2) &= (v_1 - v_2)\,g_3\,\Lambda_{1,2},\\
g_3\,g_1\,\nabla p(f_2) - g_2\,g_1\,\nabla p(f_3) &= (v_2 - v_3)\,g_1\,\Lambda_{2,3},\\
g_1\,g_2\,\nabla p(f_3) - g_3\,g_2\,\nabla p(f_1) &= (v_3 - v_1)\,g_2\,\Lambda_{3,1},
\end{aligned}$$

in $\mathcal{D}'(U,\mathbb{R}^3)$. Summing these three equations leads to

$$0 = (v_1 - v_2)\, g_3\, \Lambda_{1,2} + (v_2 - v_3)\, g_1\, \Lambda_{2,3} + (v_3 - v_1)\, g_2\, \Lambda_{3,1}, \qquad \text{in } \mathcal{D}'(U,\mathbb{R}^3).$$

Since $v_3 - v_1 = v_3 - v_2 + v_2 - v_1$, we get

$$(v_1 - v_2)\Big[\, g_3\Lambda_{1,2} - \Lambda_{3,1}\, g_2\Big] + \,(v_2 - v_3)\Big[\, g_1\Lambda_{2,3} - g_2\Lambda_{3,1}\,\Big] = 0, \qquad \text{in } \mathcal{D}'(U,\mathbb{R}^3). \quad (3.3.6)$$

For $(v_1, v_2, v_3) \in (\mathbb{R}^3)^3$, we set $V = (v_1 - v_2)|v_2 - v_3|^2 - \big[(v_1 - v_2)\cdot(v_2 - v_3)\big](v_2 - v_3)$ and

$$\begin{aligned} d(v_1 - v_2, v_2 - v_3) &= V \cdot (v_1 - v_2) \\ &= |v_1 - v_2|^2 |v_2 - v_3|^2 - \big[(v_1 - v_2)\cdot(v_2 - v_3)\big]^2. \end{aligned}$$

Easy calculations lead to the following properties of d:

Lemma 3.3.4 *For every $(v_1, v_2, v_3) \in (\mathbb{R}^3)^3$, the function d satisfies*

- $d(v_1 - v_2, v_2 - v_3) = d(v_1 - v_2, v_1 - v_3) = d(v_1 - v_3, v_1 - v_2)$,
- $d(v_1 - v_2, v_2 - v_3) \geq 0$,
- $d(v_1 - v_2, v_2 - v_3) = 0 \iff v_1 - v_2$ *and* $v_2 - v_3$ *colinear,* $\iff v_1, v_2$ *and* v_3 *are aligned points in* $\mathbb{R}^3$.

In particular, if $v_1 \neq v_2$, meas $\Big\{ v_3 \in \mathbb{R}^3 \,;\, d(v_1 - v_2, v_2 - v_3) = 0 \Big\} = 0.$

Taking test functions of the form $V\,\varphi$ with $\varphi \in \mathcal{D}(U,\mathbb{R})$ in (3.3.6), we deduce from $V \cdot (v_2 - v_3) = 0$ that

$$d(v_1 - v_2, v_2 - v_3)\Big[\, \Lambda_{1,2}\, g_3 - \Lambda_{3,1}\, g_2\Big] = 0, \qquad \text{in } \mathcal{D}'(U,\mathbb{R}). \quad (3.3.7)$$

We set $\mathcal{P} := \big\{ v \in \mathbb{R}^3 \;/\; f(v)(1 - f(v)) > 0 \big\}$. By (3.2.2) and Lemma 3.3.2, there exist $u_i \in \mathbb{R}^3$ and $r_i > 0$, $i = 1, 2, 3$ such that (3.3.3) and (3.3.4) hold. We first show that $f \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$ and $p(f) \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash B_3, \mathbb{R})$. For $i = 1, 2, 3$, there exists a non-negative function $\psi_i \in \mathcal{D}(\mathbb{R}^3, \mathbb{R})$ such that

$$B\left(u_i, \frac{r_i}{2}\right) \subset \text{ supp } (\psi_i) \subset B_i. \quad (3.3.8)$$

By (3.3.3) and the definition of $\mathcal{P}$, we have $\int g_3\, \psi_3(v_3)\, dv_3 > 0$. Owing to Lemma 3.3.4,

$$\int d(v_1 - v_2, v_2 - v_3)\, g_3\, \psi_3(v_3)\, dv_3 > 0, \qquad \forall (v_1, v_2) \in \Omega.$$

Moreover, the function

$$(v_1, v_2) \mapsto \int d(v_1 - v_2, v_2 - v_3)\, g_3\, \psi_3(v_3)\, dv_3$$

belongs to $\mathcal{C}^\infty(\Omega,\mathbb{R})$. Taking test functions of the form $(v_1-v_2)\cdot\varphi(v_1,v_2)\,\psi_3(v_3)$ with $\varphi\in\mathcal{D}(\Omega\backslash(B_3\times\mathbb{R}^3\cup\mathbb{R}^3\times B_3),\mathbb{R}^3)$ in (3.3.7) leads to

$$g_1\nabla p(f_2)-g_2\nabla p(f_1)=(v_1-v_2)\,G_{v_1}(v_2)\,g_2 \quad \text{in} \quad \mathcal{D}'(\Omega\backslash(B_3\times\mathbb{R}^3\cup\mathbb{R}^3\times B_3),\mathbb{R}^3), \tag{3.3.9}$$

where

$$G_{v_1}(v_2)=-\frac{\big\langle d(v_1-v_2,v_2-v_3)\,\Lambda_{3,1},\ \psi_3(v_3)\big\rangle_{v_3}}{\int d(v_1-v_2,v_2-v_3)\,g_3\,\psi_3(v_3)\,dv_3}.$$

We denote here by $\big\langle\ ,\ \big\rangle_{v_3}$ the dual product with respect to the v_3 variable. By (3.3.3), (3.3.8) and the definition of $\mathcal{P}$, we have $\int g_1\,\psi_1(v_1)\,dv_1>0$. Thus, taking test functions of the form $\theta(v_2)\psi_1(v_1)$ with $\theta\in\mathcal{D}(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R}^3)$ in (3.3.9), we get

$$\nabla p(f_2)=\xi(v_2)\,g_2 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^3\backslash\big(B_1\cup B_3\big),\mathbb{R}^3), \tag{3.3.10}$$

where the function ξ is defined on $\mathbb{R}^3\backslash(B_1\cup B_3)$ by

$$\xi(v_2)=\frac{1}{\big\langle g_1,\ \psi_1(v_1)\big\rangle_{v_1}}\left[\Big\langle\ \nabla p(f_1),\ \psi_1(v_1)\ \Big\rangle_{v_1}+\Big\langle\ (v_1-v_2)\,G_{v_1}(v_2),\ \psi_1(v_1)\ \Big\rangle_{v_1}\right].$$

Since $\xi\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R})$ and $g\in L^\infty(\mathbb{R}^3)$, we deduce that $p(f)\in W^{1,\infty}_{loc}(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R})$. From Sobolev embeddings follows that $p(f)\in\mathcal{C}(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R})$. We now consider $h=\sqrt{f/(1-f)}$. Since $p(f)=\mathrm{Arctan}(h)$, we deduce that $h\in\mathcal{C}(\mathbb{R}^3\backslash(B_1\cup B_3),\overline{\mathbb{R}})$. Moreover, (3.3.10) reads

$$\nabla\left(\mathrm{Arctan}(h)\right)=\xi\,\frac{h}{1+h^2} \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^3\backslash\big(B_1\cup B_3\big),\mathbb{R}^3).$$

Consequently, $\mathrm{Arctan}(h)\in\mathcal{C}^1(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R})$ and $h\in\mathcal{C}^1(\mathbb{R}^3\backslash(B_1\cup B_3),\overline{\mathbb{R}})$. By bootstrap, it follows that $h\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_1\cup B_3),\overline{\mathbb{R}})$. Thus,

$$f\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R}) \qquad \text{and} \qquad p(f)\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_1\cup B_3),\mathbb{R}).$$

The same calculations with ψ_2 instead of ψ_1 lead to $f\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_2\cup B_3),\mathbb{R})$ and $p(f)\in\mathcal{C}^\infty(\mathbb{R}^3\backslash(B_2\cup B_3),\mathbb{R})$. From (3.3.4) follows that $f\in\mathcal{C}^\infty(\mathbb{R}^3\backslash B_3,\mathbb{R})$ and $p(f)\in\mathcal{C}^\infty(\mathbb{R}^3\backslash B_3,\mathbb{R})$. The same proof with B_2 instead of B_3 implies that $f\in\mathcal{C}^\infty(\mathbb{R}^3\backslash B_2,\mathbb{R})$ and $p(f)\in\mathcal{C}^\infty(\mathbb{R}^3\backslash B_2,\mathbb{R})$. By (3.3.4), the proof of Proposition 3.3.3 is now complete. □

Proof of Theorem 3.2.3. Owing to Proposition 3.3.3, $T\in\mathcal{C}^\infty((\mathbb{R}^3)^2,\mathbb{R}^3)$. We define the real function $\overline{\Lambda}$ by

$$\overline{\Lambda}(v,v_*)=\begin{cases}(v-v_*)\cdot T/|v-v_*|^2 & \text{if } v\neq v_*\\ 0 & \text{else.}\end{cases}$$

Then, it follows from Lemma 3.3.1 that

$$T=(v-v_*)\,\overline{\Lambda}(v,v_*), \qquad \text{in } \mathcal{D}'(\Omega,\mathbb{R}^3).$$

Since T and $\overline{\Lambda}$ belong respectively to $\mathcal{C}^\infty((\mathbb{R}^3)^2, \mathbb{R}^3)$ and $\mathcal{C}^\infty(\Omega, \mathbb{R})$, this equality holds in fact a.e. on $(\mathbb{R}^3)^2$. Therefore,

$$\left.\begin{array}{rcl} g_3\, g_2\, \nabla p(f_1) - g_3\, g_1\, \nabla p(f_2) &=& (v_1 - v_2)\, \overline{\Lambda}(v_1, v_2)\, g_3, \\ g_1\, g_3\, \nabla p(f_2) - g_1\, g_2\, \nabla p(f_3) &=& (v_2 - v_3)\, \overline{\Lambda}(v_2, v_3)\, g_1, \\ g_2\, g_1\, \nabla p(f_3) - g_2\, g_3\, \nabla p(f_1) &=& (v_3 - v_1)\, \overline{\Lambda}(v_3, v_1)\, g_2, \end{array}\right. \qquad \text{a.e. on } (\mathbb{R}^3)^3.$$

As previously, we deduce that,

$$(v_1 - v_2)\, \overline{\Lambda}(v_1, v_2)\, g_3 + (v_2 - v_3)\, \overline{\Lambda}(v_2, v_3)\, g_1 + (v_3 - v_1)\, \overline{\Lambda}(v_3, v_1)\, g_2 = 0,$$

a.e. on $(\mathbb{R}^3)^3$. Consequently, multiplying by $v_2 \times v_3$ leads to

$$\det(v_1, v_2, v_3)\, \left[\, \overline{\Lambda}(v_1, v_2)\, g_3 - \overline{\Lambda}(v_3, v_1)\, g_2 \,\right] = 0 \qquad \text{a.e. on } (\mathbb{R}^3)^3.$$

Since $\operatorname{meas}\big\{(v_1, v_2, v_3) \in (\mathbb{R}^3)^3 \,;\, \det(v_1, v_2, v_3) = 0\big\} = 0$, we get

$$\overline{\Lambda}(v_1, v_2)\, g_3 - \overline{\Lambda}(v_3, v_1)\, g_2 = 0 \qquad \text{a.e. on } (\mathbb{R}^3)^3.$$

Let θ be a non-negative function from $\mathcal{D}(\mathbb{R}^3, \mathbb{R})$ such that $\int_{\mathbb{R}^3} g_2\, \theta(v_2)\, dv_2 > 0$. Then, $\overline{\Lambda}(v_3, v_1) = \mu_1\, g_3$. By symmetry, we deduce that

$$\overline{\Lambda}(v_3, v_1) = \lambda\, g_1\, g_3 \qquad \text{a.e. on } (\mathbb{R}^3)^2,$$

with $\lambda \in \mathbb{R}$. From (3.2.2) and Proposition 3.3.3 follows the existence of $u_0 \in \mathbb{R}^3$ and $r > 0$ such that $f(1-f) > 0$ on $B(u_0, r)$. Therefore,

$$f_*(1 - f_*) \nabla f - f(1-f) \nabla f_* = \lambda\, f f_* (1-f)(1-f_*)\, (v - v_*), \qquad \text{a.e. on } (B(u_0, r))^2.$$

Let ψ be a non-negative function from $\mathcal{D}(B(u_0, r), \mathbb{R})$. Then, $\int_{\mathbb{R}^3} f_*(1-f_*)\, \psi(v_*)\, dv_* > 0$. We set, if $\lambda \neq 0$,

$$\lambda\, V_0 = \frac{1}{\int_{\mathbb{R}^3} f_*\, (1 - f_*)\, \psi(v_*)\, dv_*} \Big[- \langle\, \nabla f_*\, ,\, \psi(v_*) \,\rangle + \lambda\, \langle\, f_*(1-f_*)\, v_*\, ,\, \psi(v_*) \,\rangle \Big] \in \mathbb{R}^3.$$

Then,

$$\nabla f = \lambda\, f\, (1-f)\, (v - V_0), \qquad \text{a.e. on } B(u_0, r).$$

Since $f(1-f) > 0$ on $B(u_0, r)$, we have

$$\nabla \left[\sqrt{\frac{f}{1-f}}\, e^{-\lambda \frac{|v - V_0|^2}{4}} \right] = 0, \qquad \text{on } B(u_0, r).$$

Hence,

$$f(v) = \frac{C e^{\lambda \frac{|v-V_0|^2}{2}}}{1 + C e^{\lambda \frac{|v-V_0|^2}{2}}} \qquad \text{on } B(u_0, r), \tag{3.3.11}$$

where $\lambda < 0$ because $f \in L^1(\mathbb{R}^3)$. Owing to Proposition 3.3.3, we deduce that (3.3.11) holds on $\mathbb{R}^3$. Similar calculations for $\lambda = 0$ lead to a nonintegrable function. $\square$

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Partie II

Systèmes de moments relativistes

Cette deuxième partie, composée du Chapitre 4 et de l'annexe B, est consacrée aux systèmes de moments relativistes. Dans le Chapitre 4, nous déterminons les espaces de moments relativistes qui sont compatibles avec l'invariance lorentzienne. Ce chapitre repose sur des résultats de la théorie de la représentation des groupes et des algèbres de Lie dont nous donnons une démonstration détaillée dans l'annexe B.

Chapitre 4

Moment systems derived from relativistic kinetic equations

Travail en cours en collaboration avec Pierre Degond et Mohammed Lemou.

Abstract

In this paper, we are interested in the derivation of macroscopic equations from kinetic ones using a moment method in a relativistic framework. More precisely, we establish the general form of moments that are compatible with the *Lorentz invariance* and derive a hierarchy of relativistic moment systems from a Boltzmann kinetic equation. The proof is based on the representation theory of Lie algebras. We then extend this derivation to the classical case and general families of moments that obey the *Galilean invariance* are also constructed. It is remarkable that the set of formal classical limits of the so-obtained relativistic moment systems is not identical to the set of classical moments quoted in [C. D. Levermore, Moment closure hierarchies for kinetic theories. *J. Statist. Phys.*, **83**: 1021–1065, 1996] and one could use a new physically relevant criterion to derive suitable moment systems in the classical case. Finally, the ultra-relativistic limit is considered.

4.1 Introduction

Particle systems may be modelled at many different levels (microscopic, mesoscopic or macroscopic) depending on the scale of the studied physical phenomena and on the desired degree of accuracy for its description. In many situations, the precise knowledge of some physical quantities (density, momentum, energy, viscosity, heat flux, etc) is crucial and one cannot use the standard Euler or Navier-Stokes equations to describe such quantities. On the other hand the use of more refined models as kinetic equations is too expensive in general and makes extremely slow any realistic and accurate numerical simulation. This is due to the complexity of the kinetic equation (coupling Vlasov equation with Poisson or Maxwell equations) and to the number of involved variables (one time variable plus six space-velocity coordinates). Therefore, it is necessary in general to derive more reduced models from kinetic equations which are able to describe the desired physical quantities with a sufficient degree of accuracy. This has been a challenging subject of a large number of works in the past and still stimulates many current researches.

There are mainly two approaches to derive macroscopic equations from kinetic ones. The first one consists in deriving Euler or Navier-Stokes-like equations with various expressions for the viscosity and the heat flux. This strategy supposes that the particle distribution function is close to the so-called thermodynamical equilibrium and can be expanded into successive approximations about this equilibrium according to the well known Chapman-Enskog or Hilbert procedures. The second strategy consists in directly deriving systems of equations involving the desired macroscopic quantities (mass, momentum, energy, etc), which are moments of the distribution function with respect to the velocity variable. To close the obtained systems, this strategy also needs an assumption on the distribution function which is not necessarily close to the equilibrium. For instance, Grad [8] uses an expansion in terms of Hermite polynomials whereas in [17, 18], the closure is based on the entropy minimization principle. In this last strategy, a first and important step is to derive suitable sets of moments in the velocity space, that is sets which are compatible with the *Galilean invariance* in the classical case and with the *Lorentz invariance* in the relativistic case. To our knowledge, this first step has not been solved yet.

Our work goes in the spirit of this second strategy. In this paper, we indeed establish a general form of suitable moment spaces in both the relativistic and the classical cases, that is finite dimensional spaces of polynomial functions of the energy and momentum which obey *the Lorentz* and *Galilean* invariances, respectively. The proof is based on the representation theory of Lie algebras [6, 10]. Hierarchies of moment systems have already been derived in several works in both the relativistic and the classical cases, and we refer the reader to [2, 4, 9, 13, 18] and the references therein for detailed descriptions. In these works, various closure strategies are used and the question of classical and ultra-relativistic limits is also sometimes investigated. However, the problem of deriving a general form of Lorentz or Galilean invariant sets of moments has not been addressed at a rigorous level. Our purpose here is to give a rigorous basis and a systematic way to select the families of moments that are compatible with the Lorentz invariance principle. Classical and ultra-relativistic limits of the so obtained systems are also discussed.

Before going to the presentation of our main results and for the sake of self consistency, we first recall some basic notions in relativistic mechanics. For much more detailed and complete presentations, we refer to [9, 14].

4.1.1 The relativistic kinetic model

Unlike classical mechanics where time is absolute, that is independent of the frame, a time is attached to each frame in relativistic mechanics. Therefore, the position of a particle is defined by its temporal and spatial coordinates. Let $\mathcal{R}$ and $\mathcal{R}'$ be two inertial frames such that $\mathcal{R}'$ moves with the velocity u with respect to $\mathcal{R}$. Denote by (t, x) and (t', x') the time-space coordinates respectively in $\mathcal{R}$ and $\mathcal{R}'$. Then, the change of frame is given by

$$t = \gamma_u \left(t' + \frac{u \cdot x'}{c^2} \right) \qquad \text{and} \qquad x = x' + (\gamma_u - 1)\,(u \cdot x')\,\frac{u}{|u|^2} + \gamma_u\, u\, t', \tag{4.1.1}$$

where c denotes the speed of light and

$$\gamma_u = \left(1 - \frac{|u|^2}{c^2} \right)^{-1/2}.$$

The vector $\vec{x} = (ct, x)$ is called the radius four-vector. Let $\vec{x}' = (ct', x')$ denote the radius four-vector in $\mathcal{R}'$. Then, (4.1.1) reads $\vec{x} = L_u \vec{x}'$, with

$$L_u \vec{a} = \left(\gamma_u \left(a^0 + \frac{u \cdot a}{c} \right), a + (\gamma_u - 1)\,(u \cdot a)\,\frac{u}{|u|^2} + \gamma_u\,\frac{u}{c}\,a^0 \right), \tag{4.1.2}$$

where $\vec{a} = (a^j)_{0 \le j \le 3} = (a^0, a)$ with $a = (a^j)_{1 \le j \le 3}$. The function L_u is called the proper Lorentz transformation associated to the velocity u. By analogy, any vector $\vec{y} = (y^j)_{0 \le j \le 3}$ whose components transform like those of $\vec{x}$ under a change of inertial frame is called a four-vector. An important four-vector is the energy-momentum four-vector $\vec{p} = (\varepsilon/c,\, p)$, where

$$\varepsilon = \gamma m c^2 \qquad \text{and} \qquad p = \gamma m v, \qquad \text{with} \qquad \gamma = \left(1 - \frac{|v|^2}{c^2} \right)^{-\frac{1}{2}}, \tag{4.1.3}$$

denote respectively the energy and the momentum of a relativistic particle with mass m and velocity v. It also reads

$$\varepsilon = c\sqrt{m^2 c^2 + |p|^2} \qquad \text{and} \qquad v = \frac{p}{m\sqrt{1 + \frac{|p|^2}{m^2 c^2}}}. \tag{4.1.4}$$

Similarly, any tensor of rank n whose components transform like those of the tensor product of n four-vectors under a change of inertial frame is called a four-tensor.

Let us now recall that kinetic theory generalizes to the relativistic case (see [9, 14]). During an elastic collision between two relativistic particles with momenta p and p_*, the conservation of momentum and energy holds, that is

$$p + p_* = p^\natural + p_*^\natural \qquad \text{and} \qquad \varepsilon(p) + \varepsilon(p_*) = \varepsilon(p^\natural) + \varepsilon(p_*^\natural), \tag{4.1.5}$$

where $p^\natural$ and $p_*^\natural$ denote the post-collisional momenta. As in the classical case, we may then derive the relativistic Boltzmann equation, which reads (see [7, 9, 14])

$$\partial_t f + v \cdot \nabla_x f = Q_R(f, f), \tag{4.1.6}$$

with

$$Q_R(f, f) = \iint_{\mathbb{S}^2 \times \mathbb{R}^3} \sigma(p, p_*, p^\natural, p_*^\natural)\, v_M(p, p_*)\, (f^\natural f_*^\natural - f f_*)\, dp_*\, d\omega, \tag{4.1.7}$$

where $f = f(t, x, p)$, $f_* = f(t, x, p_*)$, $f^\natural = f(t, x, p^\natural)$, $f_*^\natural = f(t, x, p_*^\natural)$, σ denotes the cross-section, v_M the Møller velocity,

$$v_M(p, p_*) = |v_{rel}|\, \frac{\varepsilon\varepsilon_* - c^2\, p \cdot p_*}{\varepsilon\varepsilon_*} = \left(|v - v_*|^2 - \frac{|v \times v_*|^2}{c^2} \right)^{1/2}, \tag{4.1.8}$$

and $d\omega$ is an element of solid angle in the centre of mass system. The structure of the relativistic Boltzmann equation (4.1.6) is similar to the classical one. Its relativistic nature appears in the relationship (4.1.4) between momentum and velocity and in the definition of the Møller velocity (4.1.8). This relativistic aspect also appears implicitly in the definition of σ, which is a non-negative function of the energy s and the deviation angle θ (in the centre of mass system), both given by

$$s = \frac{(\varepsilon + \varepsilon_*)^2}{c^2} - |p + p_*|^2,$$

and

$$\cos\theta = \frac{(\varepsilon - \varepsilon_*)\,(\varepsilon^\natural - \varepsilon_*^\natural) - c^2\,(p - p_*) \cdot (p^\natural - p_*^\natural)}{(\varepsilon - \varepsilon_*)^2 - c^2\,|p - p_*|^2}.$$

In the case of charged particles (also called the Coulomb case), the cross-section σ reads, in the centre of mass system, (see [1, Section 81, Problem 6]),

$$\sigma = \left(\frac{q q_*}{4\pi\varepsilon_0} \right)^2 \frac{1}{8c^4 (\overline{\varepsilon} + \overline{\varepsilon}_*)^2 |\overline{p}|^4 \sin^4(\theta/2)} \times \left((\overline{\varepsilon}\,\overline{\varepsilon}_* + c^2 |\overline{p}|^2)^2 + (\overline{\varepsilon}\,\overline{\varepsilon}_* + c^2 |\overline{p}|^2 \cos\theta)^2 - 2(m^2 + m_*^2) c^6 |\overline{p}|^2 \sin^2(\theta/2) \right), \tag{4.1.9}$$

where $(\overline{\varepsilon}/c, \overline{p})$ and $(\overline{\varepsilon}_*/c, \overline{p}_*)$ denote respectively the energy-momentum four-vectors $\vec{p}$ and $\vec{p}_*$ in the centre of mass system (in this system, we have $\overline{p} = -\overline{p}_*$ and then $|\overline{p}| = |\overline{p}_*|$).

Let us point out that, as for the classical Boltzmann equation, the mass, momentum and energy are locally conserved quantities for (4.1.6)-(4.1.7) and that the relativistic

Boltzmann equation possesses an entropy. In the relativistic case, the jacobian of the application $(p, p_*) \mapsto (p^\natural, p_*^\natural)$ is not equal to 1. However, since $v_M(p, p_*) = v_M(p^\natural, p_*^\natural)\, \frac{\partial(p^\natural, p_*^\natural)}{\partial(p, p_*)}$, we still have the following weak formulation,

$$\int_{\mathbb{R}^3} Q_R(f,f)\,\varphi\, dp = \frac{1}{4}\iiint_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \sigma\, v_M\, (f^\natural f_*^\natural - f f_*)(\varphi + \varphi_* - \varphi^\natural - \varphi_*^\natural)\, dp\, dp_*\, d\omega.$$

We then infer from (4.1.5) that

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ \varepsilon \end{pmatrix} f(t,x,p)\, dp,$$

are locally conserved quantities. Moreover, choosing $\varphi = \ln f$, we obtain the local dissipation law of the entropy $S(f) = \int_{\mathbb{R}^3} (f \ln f - f)\, dp$, that is

$$\partial_t S(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v\,(f \ln f - f)\, dp = \int_{\mathbb{R}^3} Q_R(f,f) \ln f\, dp \le 0. \tag{4.1.10}$$

Equilibrium states of (4.1.6) are defined to be the functions that cancel the right hand side of (4.1.10) or, equivalently, the functions $f \ge 0$ such that $Q_R(f,f) = 0$. They are the local relativistic Maxwellians

$$\mathcal{M}(p) = A \exp\big(-\beta^0\,\varepsilon(p) + \beta\cdot p\big) \qquad \text{with} \qquad A \in \mathbb{R}_+, \beta^0 \in \mathbb{R}_+,\, \beta \in \mathbb{R}^3. \tag{4.1.11}$$

4.1.2 Setting of the problem

Formally, multiplying (4.1.6) by 1, p and ε, integrating with respect to p and closing this system with the Maxwellian (4.1.11) that minimizes the entropy at fixed mass, momentum and energy, we recover the relativistic hydrodynamic equations. We are looking here for moment spaces $\mathbb{M}$ that generalize the fluid dynamic approximation and thus that contain 1, p and ε.

Moreover, the space $\mathbb{M}$ ought to respect physical symmetries. A specificity of the relativistic case is that the Galilean invariance is replaced by the Lorentzian invariance. More precisely, let L be either a proper Lorentz transformation or a rotation of the axis of the spatial coordinate system, that is L is either defined by (4.1.2) for some $u \in \mathbb{R}^3$ or given by

$$L = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix}, \tag{4.1.12}$$

where O is a 3-dimensional orthogonal matrix. Then, L corresponds either to a change of frame or to a change of axis in the momentum space. Let us denote respectively by $\vec{x}'$ and $\vec{p}'$ the radius and the energy-momentum four-vectors in the new system of coordinates. We have $\vec{x} = L^{-1}\vec{x}'$, $\vec{p} = L^{-1}\vec{p}'$,

$$\frac{dp}{\gamma(p)} = \frac{dp'}{\gamma(p')}, \qquad \sigma(p, p_*, p^\natural, p_*^\natural) = \sigma(p', p'_*, p'^\natural, p'^\natural_*), \tag{4.1.13}$$

and

$$\gamma(p)\, v_M(p,p_*)\, dp_* = |v_{rel}|\,(\varepsilon\varepsilon_* - c^2\, p\cdot p_*)\,\frac{dp_*}{m^2c^4\gamma(p_*)}$$
$$= |v_{rel}|\,(\varepsilon'\varepsilon'_* - c^2\, p'\cdot p'_*)\,\frac{dp'_*}{m^2c^4\gamma(p'_*)} = \gamma(p')\, v_M(p',p'_*)\, dp'_*, \quad (4.1.14)$$

where

$$\gamma(p) = \sqrt{1+\frac{|p|^2}{m^2c^2}}.$$

Therefore, if f denotes a solution to (4.1.6)-(4.1.7) then the function f' defined in the new system of coordinates by $f'(t',x',p') = f(t,x,p)$ is a solution to

$$\partial_{t'} f' + v(p')\cdot\nabla_{x'} f' = Q_R(f',f').$$

This corresponds to the Lorentzian invariance. The translations and the rotations that we consider in the classical case are replaced, in the relativistic case, by the proper Lorentz transformations and the rotations of the form (4.1.12). We want the space $\mathbb{M}$ to be compatible with this invariance. More precisely, let $(\varphi_1(\vec{p}),\ldots,\varphi_N(\vec{p}))$ be a moment basis for $\mathbb{M}$. Using the radius four-vector $\vec{x} = (x^j)_{0\le j\le 3}$ and the energy-momentum four-vector $\vec{p} = (p^j)_{0\le j\le 3}$, the Boltzmann equation (4.1.6) also reads

$$p^j\,\frac{\partial f}{\partial x^j} = m\,\gamma(p)\,Q_R(f,f). \quad (4.1.15)$$

Here as in the rest of this paper, we make use of the Einstein summation convention. Multiplying (4.1.15) by $\varphi_k(\vec{p})/\gamma(p)$ for some $k\in [\![1,N]\!]$ and integrating with respect to p, we obtain

$$\frac{\partial}{\partial x^j}\int_{\mathbb{R}^3} p^j\varphi_k(\vec{p})f(\vec{x},\vec{p})\frac{dp}{\gamma(p)} = m\int_{\mathbb{R}^3}\varphi_k(\vec{p})Q_R(f,f)(\vec{x},\vec{p})dp, \qquad k\in [\![1,N]\!]. \quad (4.1.16)$$

We set $\tilde{\varphi}_k = \varphi_k\circ L^{-1}$. Then, we deduce from (4.1.13) and (4.1.14) that, in the new system of coordinates, (4.1.16) reads

$$\frac{\partial}{\partial x'^j}\int_{\mathbb{R}^3} p'^j\tilde{\varphi}_k(\vec{p}\,')f'(\vec{x}\,',\vec{p}\,')\frac{dp'}{\gamma(p')} = m\int_{\mathbb{R}^3}\tilde{\varphi}_k(\vec{p}\,')Q_R(f',f')(\vec{x}\,',\vec{p}\,')dp', \qquad k\in [\![1,N]\!].$$

Consequently, a moment space $\mathbb{M}$ is said to be compatible with the Lorentzian invariance if there exist some constants $\lambda_{j,k}$ such that $\tilde{\varphi}_k = \sum_{j=1}^N \lambda_{j,k}\varphi_j$ for $k=1,\ldots,N$. We are looking here for spaces that are invariant under any proper Lorentz transformation and any rotation in the momentum space.

Moreover, as in [17], we introduce the convex cone

$$\mathbb{M}_c := \left\{ r\in\mathbb{M} : \int_{\mathbb{R}^3}\exp(r(\varepsilon(p),p))\,dp < \infty \right\},$$

for every space $\mathbb{M}$ constituted of functions of p and ε. In [17], Levermore introduced admissible moment spaces. A moment space $\mathbb{M}$ is said to be admissible if the associated cone $\mathbb{M}_c$ has a non-empty interior in $\mathbb{M}$. We are only interested in admissible spaces.

Summarizing, we are looking for finite dimensional spaces $\mathbb{M}$ of polynomial functions of the energy and momentum that satisfy

(I) $\mathrm{span}\left(1,\, p,\, \varepsilon\right) \subset \mathbb{M}$,

(II) $\mathbb{M}$ is invariant under any proper Lorentz transformation and any rotation in the momentum space,

(III) the cone $\mathbb{M}_c$ has a non-empty interior in $\mathbb{M}$.

Here as in the rest of the paper, the span notation is applied to a collection of scalars, vectors and tensors and means all linear combinations of their components.

In order to construct spaces satisfying conditions (I), (II) and (III), a first idea is to consider tensor products of the four-vector $\vec{p}$. Thus, for every $n \in \mathbb{N}_*$, we set

$$\mathcal{T}_n(\vec{p}) = \otimes^n \vec{p}, \tag{4.1.17}$$

and denote by $\mathbb{P}_n$ the space generated by the components of $\mathcal{T}_n$. We point out that each $\mathbb{P}_n$ satisfies condition (II). Since $\mathrm{span}(1,\, p,\, \varepsilon)$ is itself invariant under any proper Lorentz transformation and any rotation in the momentum space, we set, for every $n \in \mathbb{N}_*$,

$$\mathbb{M}_n = \mathrm{span}(1, p, \varepsilon, \mathcal{T}_n). \tag{4.1.18}$$

It only remains to check that condition (III) holds. Given $r \in \mathbb{M}_n$,

$$r = - \sum_{(i_1,\dots,i_n)\in[\![0,3]\!]^n} \alpha_{i_1,\dots,i_n} \mathcal{T}_n^{i_1,\dots,i_n}(\vec{p}) + \beta + \gamma \cdot p + \delta\varepsilon,$$

it suffices to suppose that the coefficient $\alpha_{0,\dots,0}$ in front of $\mathcal{T}_n^{0,\dots,0}(\vec{p}) = (\varepsilon/c)^n$ is large enough so that r belongs to $\mathbb{M}_{nc}$. Consequently, the space $\mathbb{M}_n$ is an admissible moment space and fulfils each of our requirements. Moreover, we point out that any vector sum of the spaces $\mathbb{M}_n$ also satisfies conditions (I), (II) and (III).

We notice that, contrary to the classical case where tensor products of the velocity vector are independent (up to symmetries), tensor products of the energy-momentum four-vector are not independent. Indeed, any component of $\mathcal{T}_{n-2k}$ may be written as a linear combination of components of $\mathcal{T}_n$, for every $k \in [\![0, [n/2]]\!]$, where $[x]$ denotes the integer part of x. Indeed, we have

$$\mathcal{T}_{n-2}^{i_1,\dots,i_{n-2}} = \frac{1}{m^2\, c^2} \sum_{(i,j)\in[\![0,3]\!]^2} g_{i,j}\, \mathcal{T}_n^{i_1,\dots,i_{n-2},i,j},$$

where $(i_1, \dots, i_{n-2}) \in [\![0, 3]\!]^{n-2}$, $g_{0,0} = 1$, $g_{1,1} = g_{2,2} = g_{3,3} = -1$ and $g_{i,j} = 0$ for $i \neq j$. Thus, any component of $\mathcal{T}_{n-2k}$, for $k \in [\![0, [n/2]]\!]$, belongs to $\mathbb{M}_n$.

A second idea to construct moment spaces is to consider contractions of $\mathcal{T}_n$. We denote by Q_n the contraction of $\mathcal{T}_n$ on any pair of indices, that is the tensor whose components satisfy

$$Q_n^{i_1,\dots,i_{n-2}} = \sum_{(i,j)\in[\![0,3]\!]^2} g_{i,j}\mathcal{T}_n^{i_1,\dots,i_{n-2},i,j}.$$

Since $(\varepsilon/c)^2 - |p|^2 = m^2c^2$, we deduce that $Q_n = m^2c^2\mathcal{T}_{n-2}$. Consequently, contrary to the classical case, we do not obtain any additional moment spaces with the contraction.

The point is now to check whether these spaces $\mathbb{M}_n$ and their vector sum are the only one to fulfil conditions (I), (II) and (III). When we only consider polynomial functions of p and ε with degree less or equal to 3, the spaces $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$ and $\mathbb{M}_2+\mathbb{M}_3$ are indeed the only one to fulfil conditions (I), (II) and (III), as shown in the next section. However, we exhibit, in the next section, a moment space with polynomial functions with degree less or equal to 4, that fulfils conditions (I), (II) and (III) and is strictly included in $\mathbb{M}_4$. The construction of such spaces rests on Theorem 4.2.1, which determines all the spaces that satisfy (II). Theorem 4.2.1 is a consequence of the representation theory of Lie groups and Lie algebras. Theorem 4.2.3 then states that the spaces exhibited in Theorem 4.2.1 are generated by the components of some tensors. In Section 4.3, we consider the classical limit of these relativistic moment spaces and suggest a new criterion for choosing moment spaces in the classical case. In Section 4.4, we are interested in the ultra-relativistic case. We then present the moment closure problem in Section 4.5. For the sake of completeness, the proof of Theorem 4.2.1 is given in Section 4.6. Finally, the representation theory of Lie groups and Lie algebras may also be used, in the classical case, to determine the spaces that are invariant under any rotation and this is stated in the appendix.

4.2 Moment system hierarchy and Lorentz invariance

We are looking for the finite dimensional subspaces of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ that satisfy conditions (I), (II) and (III). We consider the Minkowski space $\mathbb{R}^4$ endowed with the non-degenerate symmetric bilinear form g defined by

$$g(a,b) = a^0\,b^0 - a^1\,b^1 - a^2\,b^2 - a^3\,b^3, \qquad a,\, b \in \mathbb{R}^4.$$

The set of real matrices $L = (L_{i,j})_{0\le i,j\le 3} \in \mathcal{M}(4,\mathbb{R})$ that leave g invariant (i.e. such that $g(Lx, Ly) = g(x,y)$ for all $x, y \in \mathbb{R}^4$) forms the generalized orthogonal group $O(1,3)$. The set of matrices L from $O(1,3)$ such that $\det(L) = 1$ and $L_{00} \ge 1$ (i.e. there is no time inversion) is called the proper Lorentz group and denoted by $SO(1,3)_e$. This group is generated by two different kinds of transformations, the proper Lorentz transformations and the rotations in the momentum space. Therefore, we consider the following action of $SO(1,3)_e$ on the subspace $\mathcal{P}_n$ composed of the polynomials of $\mathbb{R}[y_0, y_1, y_2, y_3]$ with total degree less or equal to n,

$$\begin{aligned} \varphi : SO(1,3)_e &\longrightarrow GL(\mathcal{P}_n) \\ L &\longmapsto \{R(y_0,y_1,y_2,y_3) \mapsto R(L^{-1}(y_0,y_1,y_2,y_3))\}. \end{aligned} \tag{4.2.1}$$

Finding the finite dimensional subspaces of $\mathbb{R}[\varepsilon,p^1,p^2,p^3]$ that satisfy condition (II) amounts to finding the irreducible subrepresentations of $(\varphi,\mathcal{P}_n)$. This is the aim of the following theorem, which rests on the representation theory of Lie groups and Lie algebras. Its proof is postponed to Section 4.6.

Theorem 4.2.1 *A space W is an irreducible subrepresentation of $(\varphi,\mathcal{P}_n)$ if and only if there exist $j \in [\![0,[n/2]]\!]$ and some real numbers $(\lambda_k)_{0\le k\le j}$ such that W is generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k (y_0^2-y_1^2-y_2^2-y_3^2)^{j-k} \sum_{m=\max(q-r,0)}^{q} \frac{(n-2j-r)!\,r!}{(n-2j-r-m)!(r-q+m)!}\binom{q}{m} (y_0+y_3)^m(y_0-y_3)^{r-q+m}(y_1+iy_2)^{n-2j-r-m}(y_1-iy_2)^{q-m}. \tag{4.2.2}$$

for $q,r\in[\![0,n-2j]\!]$, $q+r\le n-2j$.

Here as in the rest of this paper, $[x]$ denotes the integer part of $x\in\mathbb{R}$ and $\binom{q}{m}$ stands for the binomial coefficient. This theorem describes all the irreducible representations of $(\varphi,\mathcal{P}_n)$. We deduce then all the spaces that satisfy condition (II) by replacing (y_0,y_1,y_2,y_3) with $(\varepsilon/c,p^1,p^2,p^3)$. We notice that, since $(\varepsilon/c)^2-|p|^2=m^2c^2$, the factor $(y_0^2-y_1^2-y_2^2-y_3^2)^{j-k}$ in (4.2.2) is replaced with the constant $(m^2c^2)^{j-k}$. Therefore, we have the following proposition.

Proposition 4.2.2 *For every $l\in\mathbb{N}$, let $\hat{\mathbb{M}}_l$ denote the vector space generated by the real parts and the imaginary parts of*

$$\sum_{m=\max(q-r,0)}^{q} \frac{(l-r)!\,r!}{(l-r-m)!(r-q+m)!}\binom{q}{m} (\varepsilon/c+p^3)^m(\varepsilon/c-p^3)^{r-q+m}(p^1+ip^2)^{l-r-m}(p^1-ip^2)^{q-m}, \tag{4.2.3}$$

for $q,r\in[\![0,l]\!]$, $q+r\le l$. Each $\hat{\mathbb{M}}_l$ satisfies condition (II).

Moreover, a finite dimensional subspace $\mathbb{M}$ of $\mathbb{R}[\varepsilon,p^1,p^2,p^3]$ satisfies condition (II) if and only if there exist $N\in\mathbb{N}$ and some $l_k\in\mathbb{N}$, $k=1,\dots,N$ such that $\mathbb{M}$ is the vector sum of the $\hat{\mathbb{M}}_{l_k}$, $k=1,\dots,N$.

Let us now check that, for each $l\in\mathbb{N}$, the spaces $\hat{\mathbb{M}}_l$ are generated by the components of some tensors.

Theorem 4.2.3 *Let $l \in \mathbb{N}$. For any tensor T of order l, we denote by $\overline{T}$ the symmetric part of T, that is the tensor whose components are*

$$\overline{T}^{j_1,\dots,j_l} = \frac{1}{l!} \sum_{\sigma \in \Sigma_l} T^{j_{\sigma(1)},\dots,j_{\sigma(l)}}, \qquad (j_1,\dots,j_l) \in [\![0,3]\!]^l,$$

where Σ_l denotes the symmetric group of order l.

Then, the vector space $\hat{\mathbb{M}}_l$ given by Proposition 4.2.2 is generated by the components of the tensor $S_l(\vec{p})$ defined by

$$S_l(\vec{p}) = \mathcal{T}_l(\vec{p}) + \sum_{k=1}^{[l/2]} \frac{(-m^2c^2)^k\,(l-k)!}{4^k (l-2k)!\,k!}\,\overline{\underbrace{g \otimes \ldots \otimes g}_{k \text{ times}} \otimes \mathcal{T}_{l-2k}(\vec{p})}, \tag{4.2.4}$$

where $\mathcal{T}_l$ is given by (4.1.17).

Proof. Let us denote by $\overline{\mathbb{M}}_l$ the vector space generated by the components of $S_l(\vec{p})$. Since

$$g_{i,j}\; S_l(\vec{p})^{i,j,k_1,\dots,k_{l-2}} = 0,$$

for any $(k_1,\dots,k_{l-2}) \in [\![0,3]\!]^{l-2}$, $S_l(\vec{p})$ has at most $(l+1)^2$ independent components. Thus, we deduce that $\dim \overline{\mathbb{M}}_l \leq \dim(\hat{\mathbb{M}}_l)$. But, by [3, Lemma 17.2.1], S_l is a four-tensor and therefore, $\overline{\mathbb{M}}_l$ satisfies condition (II). Moreover, the components of $S_l(\vec{p})$ are polynomials with degree l from $\mathcal{P}_l$. By Theorem 4.2.1, we conclude that $\overline{\mathbb{M}}_l = \hat{\mathbb{M}}_l$. ☐

We now write down the moment spaces that arise in (4.2.3) for $l = 1$, $l = 2$, $l = 3$ and $l = 4$. Moreover we also consider here conditions (I) and (III).

Case $l = 1$ $\qquad \hat{\mathbb{M}}_1 = \text{span}\,(\varepsilon, p^1, p^2, p^3)$.
This is the space generated by the four-vector $\vec{p}$. In order to satisfy condition (I), we add the mass and obtain the moment space $\mathbb{M}_1 = \text{span}\,(1, \vec{p})$. Whereas $\hat{\mathbb{M}}_1$ is a 4-dimensional space, $\mathbb{M}_1$ is a 5-dimensional space. As stated in Section 4.1, $\mathbb{M}_1$ satisfies condition (III). The corresponding equations read

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f\, dp + \nabla_x \cdot \int_{\mathbb{R}^3} f\, v\, dp = 0, \tag{4.2.5}$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f\, p\, dp + \nabla_x \cdot \int_{\mathbb{R}^3} f\, v \otimes p\, dp = 0, \tag{4.2.6}$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f\, \varepsilon\, dp + c^2\, \nabla_x \cdot \int_{\mathbb{R}^3} f\, p\, dp = 0. \tag{4.2.7}$$

Case $l = 2$

$$\hat{\mathbb{M}}_2 = \text{span}\,(\varepsilon p,\ (p^i p^j)_{i \neq j},\ m^2c^2 + |p|^2 + (p^1)^2,\ m^2c^2 + |p|^2 + (p^2)^2,\ m^2c^2 + |p|^2 + (p^3)^2).$$

The space $\hat{\mathbb{M}}_2$ is a 9-dimensional space. Adding 1, p and ε, we obtain the 14-dimensional space $\mathbb{M}_2 = \text{span}\,(1,\, \vec{p},\, \vec{p}\otimes\vec{p})$ which satisfies conditions (I), (II) and (III). The space $\mathbb{M}_2$ leads to the following 14-moment system

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v dp &= 0, && (4.2.8)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p dp &= 0, && (4.2.9)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon dp + c^2\nabla_x \cdot \int_{\mathbb{R}^3} f p dp &= 0, && (4.2.10)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon p dp + c^2\nabla_x \cdot \int_{\mathbb{R}^3} f p\otimes p dp &= \int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p dp, && (4.2.11)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p\otimes p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_R(f,f) p\otimes p dp. && (4.2.12)
\end{aligned}$$

Case $l = 3$

$$\begin{aligned}
\hat{\mathbb{M}}_3 = \text{span}\,\big(&\varepsilon(p^i p^j)_{i\neq j},\ p^1p^2p^3,\\
&\varepsilon(m^2c^2+|p|^2+3(p^1)^2),\ \varepsilon(m^2c^2+|p|^2+3(p^2)^2),\ \varepsilon(m^2c^2+|p|^2+3(p^3)^2),\\
&p_1(3m^2c^2+3|p|^2+(p^1)^2),\ p_1(m^2c^2+|p|^2+(p^2)^2),\ p_1(m^2c^2+|p|^2+(p^3)^2),\\
&p_2(m^2c^2+|p|^2+(p^1)^2),\ p_2(3m^2c^2+3|p|^2+(p^2)^2),\ p_2(m^2c^2+|p|^2+(p^3)^2),\\
&p_3(m^2c^2+|p|^2+(p^1)^2),\ p_3(m^2c^2+|p|^2+(p^2)^2),\ p_3(3m^2c^2+3|p|^2+(p^3)^2)\big).
\end{aligned}$$

The dimension of $\hat{\mathbb{M}}_3$ is thus 16. If we also consider the mass, momentum and energy, we obtain the space $\mathbb{M}_3 = \text{span}\,(1,\, \vec{p},\, \vec{p}\otimes\vec{p}\otimes\vec{p})$, whose dimension is 21. The space $\mathbb{M}_3$ satisfies conditions (I), (II) and (III) and leads to the moment system

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v dp &= 0, && (4.2.13)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p dp &= 0, && (4.2.14)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon dp + c^2\nabla_x \cdot \int_{\mathbb{R}^3} f p dp &= 0, && (4.2.15)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon p\otimes p dp + c^2\nabla_x \cdot \int_{\mathbb{R}^3} f p\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p\otimes p dp, && (4.2.16)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p\otimes p\otimes p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_R(f,f) p\otimes p\otimes p dp. && (4.2.17)
\end{aligned}$$

Case $l = 4$
The space $\hat{\mathbb{M}}_4$ is a 25-dimensional space. In order to satisfy condition (I), we add the mass, momentum and energy to $\hat{\mathbb{M}}_4$ and obtain

$$\begin{aligned}\text{span}(&1,\, p,\, \varepsilon,\, \varepsilon p^1p^2p^3,\, (m^2c^2+|p|^2)^2+6(m^2c^2+|p|^2)(p^3)^2+(p^3)^4,\\ &(m^2c^2+|p|^2)^2+6(m^2c^2+|p|^2)(p^1)^2+6(p^1p^3)^2-(p^3)^4,\, (p^1)^4-6(p^1p^2)^2+(p^2)^4,\\ &(p^1)^4+6(m^2c^2+|p|^2)(p^1)^2-6(m^2c^2+|p|^2)(p^2)^2-(p^2)^4,\\ &(p^1p^2)^2-(m^2c^2+|p|^2)(p^3)^2+(m^2c^2+|p|^2)(p^2)^2-(p^3p^1)^2,\\ &(m^2c^2+|p|^2)(p^1)^2+(p^1p^3)^2-(m^2c^2+|p|^2)(p^2)^2-(p^2p^3)^2,\\ &\varepsilon p^1(m^2c^2+|p|^2+(p^1)^2),\, \varepsilon p^1(m^2c^2+|p|^2+3(p^2)^2),\, \varepsilon p^1(m^2c^2+|p|^2+3(p^3)^2),\\ &\varepsilon p^2(m^2c^2+|p|^2+3(p^1)^2),\, \varepsilon p^2(m^2c^2+|p|^2+(p^2)^2),\, \varepsilon p^2(m^2c^2+|p|^2+3(p^3)^2),\\ &\varepsilon p^3(m^2c^2+|p|^2+3(p^1)^2),\, \varepsilon p^3(m^2c^2+|p|^2+3(p^2)^2),\, \varepsilon p^3(m^2c^2+|p|^2+(p^3)^2),\\ &p^1p^2(3(m^2c^2+|p|^2)+(p^1)^2),\, p^1p^2(3(m^2c^2+|p|^2)+(p^2)^2),\, p^1p^2(m^2c^2+|p|^2+(p^3)^2),\\ &p^1p^3(3(m^2c^2+|p|^2)+(p^1)^2),\, p^1p^3(m^2c^2+|p|^2+(p^2)^2),\, p^1p^3(3(m^2c^2+|p|^2)+(p^3)^2),\\ &p^2p^3(m^2c^2+|p|^2+(p^1)^2),\, p^2p^3(3(m^2c^2+|p|^2)+(p^2)^2),\, p^2p^3(3(m^2c^2+|p|^2)+(p^3)^2)),\end{aligned}$$

that is an admissible space with degree 30 whereas the system $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$ consists of 39 independent moments.

Conclusion
The spaces $\hat{\mathbb{M}}_l$ are strictly included in the spaces $\mathbb{M}_l$ defined by (4.1.18). However, when we also consider condition (I), we recover, for $l = 1, 2, 3$, the whole space $\mathbb{M}_l$. From $l = 4$ however, we obtain moment spaces that satisfy conditions (I) and (II) but that are still strictly included in $\mathbb{M}_l$. On the contrary, condition (III) has had no consequence since the spaces we considered were already admissible.

The finite dimensional subspaces of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ that satisfy condition (I), (II) and (III) are the vector sum of the spaces obtained as above for $l = 1, 2, 3, 4, \ldots$ Hence, the admissible space with maximal degree 1, 2, 3 or 4 are

$$\begin{aligned} &\text{maximal degree} = 1 && \mathbb{M} = \text{span}(1, \vec{p}),\\ &\text{maximal degree} = 2 && \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}),\\ &\text{maximal degree} = 3 && \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}),\\ & && \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}),\\ &\text{maximal degree} = 4 && \mathbb{M} = \hat{\mathbb{M}}_4 \oplus \text{span}(1, \vec{p}),\\ & && \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p}),\\ & && \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p}),\end{aligned}$$

which have respectively dimension 5, 14, 21, 30, 30, 39 and 55.

4.3 Classical limit

The classical limit consists in considering velocities v that are much smaller than the speed of light c. This amounts to let $c/v \to +\infty$. The equations should be rescaled but, for the sake of clarity, we keep our notations and let $c \to +\infty$. From (4.1.3), we deduce that

$$\varepsilon = mc^2 + \frac{m|v|^2}{2} + \frac{3m|v|^4}{8c^2} + O\left(\frac{1}{c^4}\right) \qquad \text{and} \qquad p = mv + \frac{mv|v|^2}{2c^2} + O\left(\frac{1}{c^4}\right). \quad (4.3.1)$$

It implies that

$$f(t,x,p) = \frac{1}{m^3}\, f_c(t,x,v) + O\left(\frac{1}{c^2}\right) \qquad \text{and} \qquad dp = m^3\, dv + O\left(\frac{1}{c^2}\right). \quad (4.3.2)$$

We denote here by f_c the distribution function in the classical case.

4.3.1 System $(1, \vec{p})$

We consider the classical limit of (4.2.5)-(4.2.7). With (4.3.1) and (4.3.2), equations (4.2.5) and (4.2.6) become

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv + O\left(\frac{1}{c^2}\right) &= 0, \\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv + O\left(\frac{1}{c^2}\right) &= 0. \end{aligned}$$

We thus obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv = 0, \quad (4.3.3)$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv = 0. \quad (4.3.4)$$

Moreover, equation (4.2.7) reads

$$mc^2\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv\right) + \frac{m}{2}\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv\right) + O\left(\frac{1}{c^2}\right) = 0,$$

which, with (4.3.3), leads to

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv + O\left(\frac{1}{c^2}\right) = 0,$$

that is,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv = 0.$$

We finally obtain the following system

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv &= 0,\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v\otimes v dv &= 0,\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv &= 0,\end{aligned}$$

that is the equations associated to the moment space span$(1, v, |v|^2)$. This space is invariant under any rotation and translation, and it is an admissible moment space.

4.3.2 System $(1, \vec{p}, \vec{p}\otimes\vec{p})$

We consider the classical limit of (4.2.8)-(4.2.12). We pass to the limit $c \to +\infty$ in (4.2.8)-(4.2.10) as we did for (4.2.5)-(4.2.7). With (4.3.1) and (4.3.2), equation (4.2.12) becomes

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c\, v\otimes v\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c\, v\otimes v\otimes v\, dv + O\left(\frac{1}{c^2}\right) = \frac{1}{m^2}\int_{\mathbb{R}^3} Q_R(f,f)\, p\otimes p\, dp.$$

We still need to pass to the limit in the collision kernel $Q_R(f,f)$. We deduce from (4.3.1) that (4.1.8) and (4.1.9) read

$$v_M = |v - v_*| + O\left(\frac{1}{c^2}\right) \qquad \text{and} \qquad \sigma = \sigma_c + O\left(\frac{1}{c^2}\right),$$

where

$$\sigma_c = \left(\frac{q q_*}{8\mu |v - v_*|^2 \pi\varepsilon_0}\right)^2 \frac{1}{\sin^4(\theta/2)} \qquad \text{with} \qquad \mu = \frac{m m_*}{m + m_*}.$$

Moreover, (4.3.2) implies that

$$f(p^\natural) f(p_*^\natural) - f(p) f(p_*) = \frac{1}{m^3 m_*^3}\left(f_c(v^\natural) f_c(v_*^\natural) - f_c(v) f_c(v_*)\right) + O\left(\frac{1}{c^2}\right),$$

where the velocities $v^\natural$ and $v_*^\natural$ are solutions to the conservation laws of momentum and energy

$$m v + m_* v_* = m v^\natural + m_* v_*^\natural \qquad \text{and} \qquad m|v|^2 + m_*|v_*|^2 = m|v^\natural|^2 + m_*|v_*^\natural|^2.$$

We thus obtain that

$$Q_R(f,f) = \frac{1}{m^3}\, Q_C(f_c, f_c) + O\left(\frac{1}{c^2}\right), \tag{4.3.5}$$

where Q_C denotes the classical collision kernel

$$Q_C(f_c, f_c)(t,x,v) = \iint_{\mathbb{S}^2\times\mathbb{R}^3} \sigma_c\, |v - v_*|\, (f_c(v^\natural) f_c(v_*^\natural) - f_c(v) f_c(v_*))\, dv_*\, d\omega.$$

From (4.3.5), we deduce that (4.2.12) reads

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv + O\left(\frac{1}{c^2}\right).$$

We finally get

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv = \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv.$$

Similarly to (4.2.7), equation (4.2.11) becomes

$$\begin{aligned} &m^2c^2\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c\, v\, dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c\, v\otimes v\, dv\right)\\ &+\frac{m^2}{2}\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c\, |v|^2\, v\, dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c\, |v|^2\, v\otimes v\, dv\right) + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_R(f,f)\,\varepsilon\, p\, dp. \end{aligned}$$

But,

$$\int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p dp = \frac{m^2}{2}\int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv + O\left(\frac{1}{c^2}\right),$$

whence, with (4.3.4),

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c|v|^2 v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c|v|^2 v\otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv + O\left(\frac{1}{c^2}\right).$$

Finally, system (4.2.8)-(4.2.12) becomes, letting $c\to+\infty$,

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v dv &= 0,\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v dv &= 0,\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c|v|^2 v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c|v|^2 v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv,\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv. \end{aligned}$$

We thus obtain the moment space span$(1, v, v\otimes v, v|v|^2)$, that is the Grad 13-moment system. The Grad 13-moment system is therefore compatible with the relativistic system. This space is stable under any rotation and translation. However, it is not an admissible space (in the sense of Levermore). Here the limit system has dimension 13 whereas the system (4.2.8)-(4.2.12) has dimension 14 because the equation involving $|v|^2$ is obtained twice.

4.3.3 System $(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$

We have already passed to the limit in (4.2.13)-(4.2.15). With (4.3.1), (4.3.2) and (4.3.5), equations (4.2.16) and (4.2.17) lead to

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v dv + O\left(\frac{1}{c^2}\right),$$

and

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v \otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v \otimes v dv + O\left(\frac{1}{c^2}\right).$$

Thus, letting $c \to +\infty$, we obtain the following equations

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v \otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v \otimes v dv,
\end{aligned}$$

that is the moment system corresponding to $(1, v, v \otimes v, v \otimes v \otimes v)$. This system is invariant under any rotation and translation but is not an admissible system (in the sense of Levermore). This space has dimension 20.

4.3.4 System $(1, \vec{p}, \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$

We deduce from the above calculations that passing to the limit in the moment system associated to $(1, \vec{p}, \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$ leads to

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v \otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v \otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v \otimes v \otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c, f_c) |v|^2 v \otimes v dv,
\end{aligned}$$

that is the moment system corresponding to $(1, v, v\otimes v, v\otimes v\otimes v, |v|^2 v\otimes v)$. This system is invariant under any rotation and translation. Moreover, it is an admissible system whose dimension is 26.

4.3.5 System associated to $\hat{\mathbb{M}}_4$

The system associated to $\hat{\mathbb{M}}_4$ consists of 30 independent moments. Consequently, we do not write down all the equations and we do not give all the details of the passage to the limit. The different steps are described below.

- As previously, the relativistic moment system $(1, p, \varepsilon)$ leads to the classical moment system $(1, v, |v|^2)$.
- From the set of moments of the form $p^i p^j(3(m^2c^2 + |p|^2) + (p^i)^2)$ and $p^i p^j(m^2c^2 + |p|^2 + (p^k)^2)$, we obtain the moments $v_i v_j$, $v_i v_j(v_i^2 - v_j^2)$ and $v_i v_j(v_i - 3v_k)$ for $i \neq j$, $k \neq i$ and $k \neq j$.
- Passing to the limit in the set of moments of the form $\varepsilon p^i(m^2c^2 + |p|^2 + (p^i)^2)$ and $\varepsilon p^i(m^2c^2 + |p|^2 + 3(p^k)^2)$, we get the moments $|v|^2 v_i$ and $v_i(v_i^2 - 3v_k^2)$ for $i \neq k$.
- The moment $\varepsilon p^1 p^2 p^3$ leads to the moment $v_1 v_2 v_3$.
- The six remaining moments lead to the moments $v_i^2 - v_j^2$ and $v_i^4 + 3v_j^2 v_k^2 - 3v_i^2 v_j^2 - 3v_i^2 v_k^2$, for $i \neq j$, $i \neq k$, and $j \neq k$.

Summarizing, the limit system is the following 29-dimensional space

$$\begin{aligned}\text{span}\,(&1, v, v\otimes v, v\otimes v\otimes v,\ v_1v_2(v_1^2 - v_2^2),\ v_1v_2(v_1^2 - 3v_3^2),\ v_1v_3(v_1^2 - v_3^2),\\ &v_1v_3(v_1^2 - 3v_2^2),\ v_2v_3(v_2^2 - v_3^2),\ v_2v_3(v_2^2 - 3v_1^2),\ v_1^4 + 3v_2^2v_3^2 - 3v_1^2v_2^2 - 3v_1^2v_3^2,\\ &v_2^4 + 3v_1^2v_3^2 - 3v_2^2v_3^2 - 3v_1^2v_2^2,\ v_3^4 + 3v_1^2v_2^2 - 3v_1^2v_3^2 - 3v_2^2v_3^2),\end{aligned}$$

which also reads

$$\text{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v - 6|v|^2/7\overline{I_3\otimes v\otimes v} + 3|v|^4/35 I_3\otimes I_3),$$

where $\overline{T}$ denotes the symmetric part of the tensor T. The obtained space is invariant under any rotation and any translation but it is non admissible.

4.3.6 System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$

Passing to the limit $c \to +\infty$ as in the previous sections, we get the moment space

$$\text{span}\,(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v),$$

whose dimension is 35. This space is stable under any rotation and translation and it is admissible.

4.3.7 System $(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p})$

Letting $c \to +\infty$, we get the moment space

$$\mathrm{span}\,(1, v, v \otimes v, v \otimes v \otimes v, v \otimes v \otimes v \otimes v, |v|^2 v \otimes v \otimes v).$$

This space is non admissible and has dimension 45. It is invariant under any rotation and any translation.

4.3.8 Conclusion

Passing to the limit $c \to +\infty$ on admissible relativistic systems, we have obtained admissible classical spaces as $(1, v, |v|^2)$ and $(1, v, v \otimes v, v \otimes v \otimes v, |v|^2 v \otimes v)$, but also non admissible spaces as $(1, v, v \otimes v, v|v|^2)$ and $(1, v, v \otimes v, v \otimes v \otimes v)$. Let us summarize the obtained limit systems:

Relativistic system	Limit system
$(1, \vec{p})$	$(1, v, \|v\|^2)$
$(1, \vec{p}, \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, \|v\|^2 v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, \|v\|^2 v \otimes v)$
$\hat{\mathbb{M}}_4 \oplus \mathrm{span}(1, \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v,$
	$v \otimes v \otimes v \otimes v - 6\|v\|^2/7\overline{I_3 \otimes v \otimes v} + 3\|v\|^4/35 I_3 \otimes I_3)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, v \otimes v \otimes v \otimes v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, v \otimes v \otimes v \otimes v, \|v\|^2 v \otimes v \otimes v)$

By letting $c \to +\infty$ in the relativistic moment spaces, we do not recover all the classical moment spaces, for instance, we did not get $(1, v, v \otimes v)$. Since classical mechanics is considered as an approximation of relativistic mechanics as $c \to +\infty$, it could be sensible to choose as moment spaces in the classical case only the admissible moment spaces that can be obtained as a limit of relativistic ones.

4.4 The ultra-relativistic case

The ultra-relativistic limit corresponds to the case when the total energy ε of a particle is much larger than its rest energy $m\,c^2$. As for the classical limit, the equations should be rescaled but we do not want to get the reader confused with non necessary details. Consequently, we keep our notations and let m tend to 0. Formulas (4.1.4) read

$$\varepsilon = c|p| + O(m^2) \qquad \text{and} \qquad v = c\frac{p}{|p|} + O(m^2). \tag{4.4.1}$$

We deduce then that (4.1.8) and (4.1.9) read

$$v_M = v_{M\,ur} + O\left(m^2\right) \qquad \text{and} \qquad \sigma = \sigma_{ur} + O\left(m^2\right),$$

where

$$v_{Mur} = |v_{rel}| \left(1 - \frac{p \cdot p_*}{|p||p_*|}\right) \qquad \text{and} \qquad \sigma_{ur} = \left(\frac{qq_*}{4\pi\varepsilon_0}\right)^2 \frac{1 + \cos^4(\theta/2)}{8c^2|\overline{p}|^2 \sin^4(\theta/2)}.$$

We thus obtain that

$$Q_R(f,f) = Q_{ur}(f_{ur}, f_{ur}) + O\left(m^2\right), \tag{4.4.2}$$

where f_{ur} denotes the ultra-relativistic distribution function and Q_{ur} the ultra-relativistic collision kernel

$$Q_{ur}(f_{ur}, f_{ur})(t,x,v) = \iint_{\mathbb{S}^2\times\mathbb{R}^3} v_{Mur}\sigma_{ur}\left(f_{ur}(p^\natural)f_{ur}(p_*^\natural) - f_{ur}(p)f_{ur}(p_*)\right) dp_*\, d\omega.$$

System $(1, \vec{p})$
With (4.4.1), equations (4.2.5)-(4.2.7) lead to

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp &= 0, &(4.4.3)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}pdp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes pdp &= 0, &(4.4.4)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}pdp &= 0, &(4.4.5)
\end{aligned}$$

that is the 5-moment system associated to $(1, p, |p|)$.

System $(1, \vec{p}, \vec{p}\otimes\vec{p})$
By (4.4.1) and (4.4.2), equations (4.2.8)-(4.2.12) read

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}pdp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes pdp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}pdp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|pdp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}p\otimes pdp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur})|p|pdp,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p\otimes pdp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes p\otimes pdp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur})p\otimes pdp.
\end{aligned}$$

We thus obtain the 14-dimensional system $(1, p, |p|, p|p|, p\otimes p)$.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p})$
In the ultra-relativistic case, equations (4.2.13)-(4.2.17) become

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp = 0,$$

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur} p dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p\otimes p}{|p|}dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p| dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur} p dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p| p\otimes p dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur} p\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur})|p| p\otimes p dp,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur} p\otimes p\otimes p dp + c\nabla_x \cdot \int_{\mathbb{R}^3} f_{ur}\frac{p\otimes p\otimes p\otimes p}{|p|}dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur}) p\otimes p\otimes p dp.
\end{aligned}$$

We thus obtain the system $(1, p, |p|, |p|p\otimes p, p\otimes p\otimes p)$, whose dimension is 21.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p})$
Letting $m \to 0$, we obtain the moment space

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p,\ p\otimes p\otimes p),$$

whose dimension is 30.

System associated to $\hat{\mathbb{M}}_4$
In the ultra-relativistic case, the system obtained from $\hat{\mathbb{M}}_4$ by adding the mass, momentum and energy reads

$$\begin{aligned}
\text{span}(&1,\ p,\ |p|,\ |p|p^1p^2p^3,\ |p|^4+6|p|^2(p^3)^2+(p^3)^4,\ |p|^4+6|p|^2(p^1)^2+6(p^1p^3)^2-(p^3)^4,\\
&(p^1p^2)^2-|p|^2(p^3)^2+|p|^2(p^2)^2-(p^3p^1)^2,\ |p|^2(p^1)^2+(p^1p^3)^2-|p|^2(p^2)^2-(p^2p^3)^2,\\
&(p^1)^4-6(p^1p^2)^2+(p^2)^4,\ (p^1)^4+6|p|^2(p^1)^2-6|p|^2(p^2)^2-(p^2)^4,\ |p|(p^1)^3,\ |p|p^1(p^2)^2,\\
&|p|p^1(p^3)^2,\ |p|p^2(p^1)^2,\ |p|(p^2)^3,\ |p|p^2(p^3)^2,\ |p|p^3(p^1)^2,\ |p|p^3(p^2)^2,\ |p|(p^3)^3,\ (p^1)^3p^2,\\
&p^1(p^2)^3,\ p^1p^2(p^3)^2,\ (p^1)^3p^3,\ p^1(p^2)^2p^3,\ p^1(p^3)^3,\ (p^1)^2p^2p^3,\ (p^2)^3p^3,\ p^2(p^3)^3).
\end{aligned}$$

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$
Passing to the limit $m \to 0$, we get the moment space

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p\otimes p,\ p\otimes p\otimes p\otimes p),$$

whose dimension is 39.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$
In the ultra-relativistic case, the moment space $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$ leads to

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p,\ p\otimes p\otimes p,\ |p|p\otimes p\otimes p,\ p\otimes p\otimes p\otimes p).$$

This space has dimension 55.

4.5 Moment closure problem

4.5.1 The maximum entropy principle

Up to now, we have determine the moment spaces $\mathbb{M}$ that could be used to derive moment system in both the relativistic and the ultra-relativistic cases. The moment system is then obtained by multiplying the Boltzmann equation by a basis $m = (m_i)_{1\leq i\leq N}$ of $\mathbb{M}$ and integrating with respect to the momentum variable. However, the obtained moment system is not closed unless a distribution function is specified. A usual strategy consists in closing this system using the function that solves the entropy minimization problem. Given $M \in \mathbb{R}^N$, we close the system using the distribution f that realizes the following minimum

$$\min\left\{S(f) = \int_{\mathbb{R}^3} (f\ln f - f)\,dp, \qquad \int_{\mathbb{R}^3} f(p)\,m(p,\varepsilon(p))\,dp = M\right\}. \tag{4.5.1}$$

It is of course not warranted that this problem has a solution. Indeed, the vector $M \in \mathbb{R}^N$ needs to satisfy some constraints in order that there exists a distribution f such that

$$\int_{\mathbb{R}^3} f(p)\,m(p,\varepsilon(p))\,dp = M. \tag{4.5.2}$$

Formally, the method of the Lagrange multipliers imply that the solution f (if it exists) to the entropy minimization problem (4.5.1) satisfies

$$f(t,x,p) = \exp\big(\alpha(t,x)\cdot m(p,\varepsilon(p))\big),$$

where the coefficient $\alpha(t,x) \in \mathbb{R}^N$ is uniquely determined by the constraint (4.5.2). Consequently, one important point is to know whether there exist exponential densities that satisfy (4.5.2). This problem has been studied in the classical case by Junk [11], who assumed that there exists one moment of the basis $(m_i)_{1\leq i\leq N}$ that grows faster than the others at infinity. This assumption is fulfilled neither by the relativistic moment spaces nor by the ultra-relativistic moment spaces. It would be interesting to see if the results of Junk could however be extended to these cases. Of course, the ultra-relativistic case should be easier to handle because the corresponding energy $\overline{\varepsilon} = c|p|$ is much simpler than the relativistic one. Consequently, some calculations might be explicit. This problem is not considered herein, and is the subject of a future work.

In the relativistic and ultra-relativistic cases, the moment realizability problem has already been solved for the moments $(1, p, \varepsilon(p))$ and $(1, p, \overline{\varepsilon}(p))$. Consequently, for these moment systems, the closure by the entropy minimization principle can be carried out. We summarize the main ideas below.

4.5.2 The relativistic case

We proceed here below to the closure of the 5 moment system (4.2.5)-(4.2.7). By [7, Theorem 3.15.3] and [5, Theorem 2.1], there exists a solution to the problem of minimizing the entropy at fixed mass n, momentum P and energy W if and only if n, P and W satisfy $m^2c^2n^2 + |P|^2 \leq W^2/c^2$ and this solution is uniquely determined. This solution is the relativistic Maxwellian of the form (4.1.11) that satisfies

$$n = \int_{\mathbb{R}^3} \mathcal{M}(p)\,dp, \qquad P = \int_{\mathbb{R}^3} \mathcal{M}(p)\,p\,dp \qquad \text{and} \qquad W = \int_{\mathbb{R}^3} \mathcal{M}(p)\,\varepsilon(p)\,dp.$$

The closure of the system (4.2.5)-(4.2.7) by this Maxwellian enables us to compute the fluxes $\int_{\mathbb{R}^3} f v dp$ and $\int_{\mathbb{R}^3} f v \otimes p dp$ in terms of n, P and W. We then obtain the relativistic hydrodynamic equations (see [13, 15])

$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ P \\ W \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} n\,u \\ P \otimes u + \mathcal{P}\,\gamma_u^{-1}\,Id \\ W\,u + \mathcal{P}\,\gamma_u^{-1}\,u \end{pmatrix} = 0,$$

where $\mathcal{P}$, W and P are related to T, u and n by

$$\mathcal{P} = n\,k\,T, \qquad W = \gamma_u\left(n\,e_0(T) + \frac{|u|^2}{c^2}\mathcal{P}\right) \qquad \text{and} \qquad P = \gamma_u\,\frac{u}{c^2}\,\big(n\,e_0(T) + \mathcal{P}\big).$$

Here, we denote respectively by u, $\mathcal{P}$, T and e_0 the velocity, the stress, the temperature and the proper internal energy of the fluid. The constant k is Boltzmann's constant.

4.5.3 The ultra-relativistic case

We close here the system (4.4.3)-(4.4.5) thanks to the Maxwellian that minimizes the entropy at fixed mass $\tilde{n}$, momentum $\tilde{P}$ and energy $\tilde{W}$. The proof of [5, Theorem 2.1] can be extended to the ultra-relativistic case and there exists a solution to this problem if and only if $\tilde{P}$ and $\tilde{W}$ satisfy $|\tilde{P}| \leq \tilde{W}/c$, this solution being uniquely determined. This solution is the ultra-relativistic Maxwellian of the form

$$\tilde{\mathcal{M}}(p) = A\,\exp\big(-\beta^0\,|p| + \beta\cdot p\big) \qquad \text{with} \qquad A \in \mathbb{R}_+, \beta^0 \in \mathbb{R}_+,\ \beta \in \mathbb{R}^3,$$

that satisfies

$$\tilde{n} = \int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\,dp, \qquad \tilde{P} = \int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\,p\,dp \qquad \text{and} \qquad \tilde{W} = c\int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\,|p|\,dp.$$

We obtain the ultra-relativistic hydrodynamic equations (see [12])

$$\frac{\partial}{\partial t}\begin{pmatrix} \tilde{n} \\ \tilde{P} \\ \tilde{W} \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} \tilde{n}\,\tilde{u} \\ \tilde{P} \otimes \tilde{u} + \tilde{\mathcal{P}}\,\gamma_{\tilde{u}}^{-1}\,Id \\ \tilde{W}\,\tilde{u} + \tilde{\mathcal{P}}\,\gamma_{\tilde{u}}^{-1}\,\tilde{u} \end{pmatrix} = 0,$$

where $\tilde{\mathcal{P}}$, $\tilde{W}$ and $\tilde{P}$ are related to $\tilde{T}$, $\tilde{u}$ and $\tilde{n}$ by

$$\tilde{\mathcal{P}} = \tilde{n}\,k\,\tilde{T}, \qquad \tilde{W} = \gamma_{\tilde{u}}\left(3 + \frac{|\tilde{u}|^2}{c^2}\right)\tilde{\mathcal{P}} \qquad \text{and} \qquad \tilde{P} = 4\,\gamma_{\tilde{u}}\,\frac{\tilde{u}}{c^2}\,\tilde{\mathcal{P}}.$$

Here, we denote respectively by $\tilde{u}$, $\tilde{\mathcal{P}}$ and $\tilde{T}$ the velocity, the stress and the temperature of the fluid. The constant k is Boltzmann's constant.

4.6 Proof of Theorem 4.2.1

This section is based on the representation theory of Lie groups and Lie algebras. Therefore, we refer to [6, 10] for further information. The group $SO(1,3)_e$ is a matrix Lie group. We point out that Lie algebras are essential for the study of matrix Lie groups because they have the advantage of being vector spaces and thus allow the use of linear algebra tools. The Lie algebra associated to $SO(1,3)_e$ reads

$$so_{\mathbb{R}}(1,3) = \left\{ X \in M(4,\mathbb{R}); gX^T + Xg = 0 \right\},$$

where X^T denotes the matrix transpose of X. Let $so_{\mathbb{C}}(1,3)$ be the complexification of $so_{\mathbb{R}}(1,3)$ (see [10, Definition 2.43]),

$$so_{\mathbb{C}}(1,3) = \left\{ X \in M(4,\mathbb{C}); gX^T + Xg = 0 \right\}.$$

We denote by $\mathbb{C}_n[y_0,y_1,y_2,y_3]$ the set of complex homogeneous polynomials with degree n and consider the following representation of $SO(1,3)_e$:

$$\begin{array}{rcl} \tilde{\varphi} : SO(1,3)_e & \longrightarrow & GL(\mathbb{C}_n[y_0,y_1,y_2,y_3]) \\ L & \longmapsto & \{R(y_0,y_1,y_2,y_3) \longmapsto R(L^{-1}(y_0,y_1,y_2,y_3))\} \end{array}$$

By [10, Proposition 4.4], the representation $(\tilde{\varphi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $SO(1,3)_e$ induces a unique representation $(\Phi, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{R}}(1,3)$, which is defined by

$$\Phi(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ})\Big|_{t=0}, \qquad Z \in so_{\mathbb{R}}(1,3).$$

By [10, Proposition 4.6], this finite dimensional complex representation of $so_{\mathbb{R}}(1,3)$ may be uniquely extended to a complex representation $(\tilde{\Phi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{C}}(1,3)$, given by

$$\tilde{\Phi}(Z) = \Phi(Z_1) + i\Phi(Z_2), \qquad Z = Z_1 + iZ_2 \in so_{\mathbb{C}}(1,3), \quad Z_1, Z_2 \in so_{\mathbb{R}}(1,3).$$

The representation theory of Lie algebras implies, thanks to the highest weight theory, that the following theorem holds:

Theorem 4.6.1 *The representation $(\tilde{\Phi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{C}}(1,3)$ is not irreducible. More precisely, we have the following decomposition:*

$$\mathbb{C}_n[y_0,y_1,y_2,y_3] = \bigoplus_{j=0}^{[n/2]} \Gamma^{(n)}_{n-2j,0}, \tag{4.6.1}$$

where $\Gamma^{(n)}_{n-2j,0}$ is a subspace of $\mathbb{C}_n[y_0,y_1,y_2,y_3]$ generated by

$$\Bigg((y_0^2 - y_1^2 - y_2^2 - y_3^2)^j \sum_{m=\max(l-k,0)}^{\min(l,n-2j-k)} \frac{(n-2j-k)!}{(n-2j-k-m)!} \frac{k!}{(k-l+m)!} \binom{l}{m} (y_1 - iy_2)^{n-2j-k-m} (y_0+y_3)^m (y_0-y_3)^{k-l+m} (y_1+iy_2)^{l-m} \Bigg)_{0 \le k,\, l \le n-2j}. \tag{4.6.2}$$

Each representation $(\tilde{\Phi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ is irreducible.

It then follows easily that

Theorem 4.6.2 *The representation* $(\tilde{\varphi}, \mathbb{C}_n[y_0, y_1, y_2, y_3])$ *of* $SO(1,3)_e$ *is not irreducible. More precisely, (4.6.1) holds and each* $(\tilde{\varphi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ *is an irreducible representation.*

Proof. We have to show that each $\Gamma^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$. Since $SO(1,3)_e$ is generated by the matrices

$$R_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}, \qquad R_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & 0 & \sin t \\ 0 & 0 & 1 & 0 \\ 0 & -\sin t & 0 & \cos t \end{pmatrix},$$

$$R_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L_1(t) = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L_2(t) = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3(t) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix},$$

for $t \in \mathbb{R}$, it suffices to show that the vector space $\Gamma^{(n)}_{n-2j,0}$ is stable under any $\tilde{\varphi}(R_k(t))$ and $\tilde{\varphi}(L_k(t))$, for every $t \in \mathbb{R}$. The space $so_{\mathbb{C}}(1,3)$ is generated by

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.6.3}$$

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.6.4}$$

Consequently, each $\Gamma^{(n)}_{n-2j,0}$ is stable under $\Phi(R_k)$ and $\Phi(L_k)$, $1 \leq k \leq 3$. But, we have

$$R_k(t) = \exp(tR_k) \qquad \text{and} \qquad L_k(t) = \exp(tL_k), \qquad t \in \mathbb{R},$$

and, by [10, Proposition 4.4],

$$\tilde{\varphi}\left(e^X\right) = e^{\Phi(X)}, \qquad X \in so_{\mathbb{R}}(1,3).$$

We thus deduce that $\Gamma^{(n)}_{n-2j,0}$ is stable under any $\tilde{\varphi}(R_k(t))$ and $\tilde{\varphi}(L_k(t))$, for every $t \in \mathbb{R}$.

Since $SO(1,3)_e$ is a connected matrix Lie group, the representation $(\tilde{\varphi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ of $SO(1,3)_e$ is irreducible by [10, Proposition 4.5] and [10, Proposition 4.6]. □

It remains now to consider the case of real polynomials. We denote by $\mathbb{R}_n[y_0, y_1, y_2, y_3]$ the set of real homogeneous polynomials with degree n and consider the representation $(\varphi_{|\mathbb{R}_n[y_0,y_1,y_2,y_3]}, \mathbb{R}_n[y_0, y_1, y_2, y_3])$ of $SO(1,3)_e$, where φ is defined by (4.2.1).

Theorem 4.6.3 *The representation $(\varphi_{|\mathbb{R}_n[y_0,y_1,y_2,y_3]}, \mathbb{R}_n[y_0, y_1, y_2, y_3])$ of $SO(1,3)_e$ is not irreducible. We have the following decomposition:*

$$\mathbb{R}_n[y_0, y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \tilde{\Gamma}^{(n)}_{n-2j,0}, \tag{4.6.5}$$

where the space $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is generated by the real parts and the imaginary parts of

$$(y_0^2 - y_1^2 - y_2^2 - y_3^2)^j \sum_{m=\max(q-r,0)}^{q} \frac{(n-2j-r)!r!}{(n-2j-r-m)!(r-q+m)!}\binom{q}{m} (y_0+y_3)^m (y_0-y_3)^{r-q+m} (y_1+iy_2)^{n-2j-r-m} (y_1-iy_2)^{q-m}, \tag{4.6.6}$$

for $q, r \in [\![0, n-2j]\!]$, $q + r \leq n - 2j$. The subrepresentations $(\varphi_{|\tilde{\Gamma}^{(n)}_{n-2j,0}}, \tilde{\Gamma}^{(n)}_{n-2j,0})$ are irreducible.

Proof. Let $(q, r) \in \mathbb{N}^2$ such that $q + r < n - 2j$. Choosing $(k, l) = (r, q)$ and $(k, l) = (n - 2j - q, n - 2j - r)$, we deduce that the complex basis (4.6.2) can be replaced by the real parts and imaginary parts of (4.6.6). We denote by $\tilde{\Gamma}^{(n)}_{n-2j,0}$ the real vector space generated by the real parts and the imaginary parts of (4.6.6). It follows from (4.6.1) that (4.6.5) holds.

We still have to check that each $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$. By Theorem 4.6.2, we already know that the complexification $\Gamma^{(n)}_{n-2j,0}$ of $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$. Since the proper Lorentz group $SO(1,3)_e$ is composed of real matrices, we deduce that the real vector space $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$.

Each representation $(\varphi_{|\tilde{\Gamma}^{(n)}_{n-2j,0}}, \tilde{\Gamma}^{(n)}_{n-2j,0})$ of $SO(1,3)_e$ is irreducible. Indeed, if there is a subspace V of $\tilde{\Gamma}^{(n)}_{n-2j,0}$ stable under $SO(1,3)_e$, then the complexification $V + iV$ of V is a subspace of $\Gamma^{(n)}_{n-2j,0}$ stable under $SO(1,3)_e$. Since $\Gamma^{(n)}_{n-2j,0}$ is irreducible, we deduce that either $V = \{0\}$, or $V = \tilde{\Gamma}^{(n)}_{n-2j,0}$. □

We now consider the representation (4.2.1) of $SO(1,3)_e$ and we prove Theorem 4.2.1. It follows from (4.6.5) that we have the following decomposition of $(\varphi, \mathcal{P}_n)$ as a direct sum of irreducible representations

$$\mathcal{P}_n = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{[k/2]} \tilde{\Gamma}^{(k)}_{k-2j,0}, \tag{4.6.7}$$

where $\tilde{\Gamma}^{(k)}_{k-2j,0}$ is the real vector space given by Theorem 4.6.3. The Schur Lemma (see [10, Theorem 4.26]) implies that this decomposition as a direct sum is unique up to an isomorphism. Its proof follows the same lines as [16, Proposition 1.2] for modules. We now need the following lemma:

Lemma 4.6.4 *Let $\hat{\mathcal{P}}_n = \oplus_{k=0}^n \mathbb{C}_k[y_0, y_1, y_2, y_3]$. We consider the following representation:*

$$\begin{array}{rcl} \hat{\varphi} : SO(1,3)_e & \longrightarrow & GL(\hat{\mathcal{P}}_n) \\ L & \longmapsto & \{R(y_0, y_1, y_2, y_3) \longmapsto R(L^{-1}(y_0, y_1, y_2, y_3))\} \end{array}$$

Denote by $(\hat{\Phi}, \hat{\mathcal{P}}_n)$ the associated representation of $so_{\mathbb{C}}(1,3)$. A non-zero polynomial $Q \in \hat{\mathcal{P}}_n$ is said to be a highest weight vector associated to the weight $(n-2j, 0)$ of the representation $(\hat{\Phi}, \hat{\mathcal{P}}_n)$ of $so_{\mathbb{C}}(1,3)$ if

$$\hat{\Phi}(D_1)Q = (n-2j)Q, \qquad \hat{\Phi}(D_2)Q = 0, \qquad \hat{\Phi}(C_1)Q = 0, \qquad \hat{\Phi}(C_3)Q = 0, \qquad (4.6.8)$$

with $D_1 = iR_3$, $D_2 = L_3$, $C_1 = R_1 + L_2 + i(R_2 + L_1)$ and $C_3 = R_1 - L_2 + i(R_2 - L_1)$, where the matrices R_j and L_j are defined by (4.6.3) and (4.6.4).

A polynomial $Q \in \hat{\mathcal{P}}_n$ satisfies (4.6.8) if and only if

$$Q(y_0, y_1, y_2, y_3) = (y_1 - iy_2)^{n-2j} \sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} \qquad (4.6.9)$$

with $\lambda_k \in \mathbb{C}$ for $k = 0, \ldots, j$.

Proof. Let us assume that $Q \in \hat{\mathcal{P}}_n$ satisfies (4.6.8). As previously, by [10, Proposition 4.4] and [10, Proposition 4.6], we have

$$\hat{\Phi}(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ_1})\Big|_{t=0} + i\frac{d}{dt}\tilde{\varphi}(e^{tZ_2})\Big|_{t=0}, \qquad Z = Z_1 + iZ_2, \qquad Z_1, Z_2 \in so_{\mathbb{R}}(1,3).$$

The change of variables $Y = PX$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

diagonalizes both D_1 and D_2 and implies that the coefficients of

$$\tilde{Q}(x_0, x_1, x_2, x_3) = \sum_{k_0+k_1+k_2+k_3 \leq n} a_{k_0,k_1,k_2,k_3}\, x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3},$$

where $\tilde{Q}(X) = Q(PX)$, satisfy

$$\begin{array}{rcll} (n-2j-k_1+k_2)a_{k_0,k_1,k_2,k_3} & = & 0, & (4.6.10) \\ (k_3-k_0)a_{k_0,k_1,k_2,k_3} & = & 0, & (4.6.11) \\ (k_2+1)a_{k_0-1,k_1,k_2+1,k_3} + (k_3+1)a_{k_0,k_1-1,k_2,k_3+1} & = & 0, & k_0 \geq 1, k_1 \geq 1, \ (4.6.12) \\ (k_0+1)a_{k_0+1,k_1-1,k_2,k_3} + (k_2+1)a_{k_0,k_1,k_2+1,k_3-1} & = & 0, & k_1 \geq 1, k_3 \geq 1. \ (4.6.13) \end{array}$$

for every $(k_0, k_1, k_2, k_3) \in \mathbb{N}^4$, $k_0 + k_1 + k_2 + k_3 \leq n$. We thus deduce that

$$\tilde{Q}(x_0, x_1, x_2, x_3) = \sum_{l=0}^{j} \sum_{m=0}^{j-l} a_{m,n-j-m-l,j-m-l,m}\, x_0^m\, x_1^{n-j-m-l}\, x_2^{j-m-l}\, x_3^m.$$

Let $l_0 \in [\![0, j]\!]$ and $m_0 \in [\![0, j-l]\!]$ be such that $a_{m_0,n-j-m_0-l_0,j-m_0-l_0,m_0} \neq 0$. Equations (4.6.12) and (4.6.13) imply that

$$a_{m,n-j-m-l,j-m-l,m} = (-1)^{j-m-l} \binom{j-l}{m} \frac{a_{m_0,n-j-m_0-l_0,j-m_0-l_0,m_0}(-1)^{j-m_0-l_0}}{\binom{j-l_0}{m_0}},$$

for each $l \in [\![0, j]\!]$ and $m \in [\![0, j-l]\!]$. Consequently,

$$\tilde{Q}(x_0, x_1, x_2, x_3) = x_1^{n-2j} \sum_{l=0}^{j} \lambda_l (x_0 x_3 - x_1 x_2)^{j-l},$$

with $\lambda_l \in \mathbb{C}$ for $l = 0, \ldots, j$. $\square$

Proof of Theorem 4.2.1. Let W be an irreducible subrepresentation of $(\varphi, \mathcal{P}_n)$.

Let us recall that, by [10, Proposition 4.33], for every finite dimensional representation (Π, V) of a Lie group that decomposes as a direct sum of irreducible representations, every stable subspace of V also decomposes as a direct sum of irreducible representations and, given a stable subspace U of V, there is a stable subspace $\tilde{U}$ such that $V = U \oplus \tilde{U}$.

Case 1: There exists $k \in [\![0, n]\!]$ such that $W \subset \mathbb{R}_k[y_0, y_1, y_2, y_3]$.

It follows from (4.6.5) and [10, Proposition 4.33] that there exists a stable subspace $W' \subset \mathbb{R}_k[y_0, y_1, y_2, y_3]$ such that $\mathbb{R}_k[y_0, y_1, y_2, y_3] = W \oplus W'$. Then, [10, Proposition 4.33] implies that $W' = \oplus_\alpha \Gamma_\alpha$ and, thus,

$$\mathbb{R}_k[y_0, y_1, y_2, y_3] = W \bigoplus \oplus_\alpha \Gamma_\alpha.$$

By uniqueness of (4.6.5), there exists $j \in [\![0, [k/2]]\!]$ such that $W \simeq \tilde{\Gamma}^{(k)}_{k-2j,0}$. Considering the complexification $W_{\mathbb{C}} = W + iW$ of W and extending the action of $SO(1,3)_e$ on $W_{\mathbb{C}}$ to an action of $so_{\mathbb{C}}(1,3)$, we deduce from [10, Proposition 7.15] that $W_{\mathbb{C}}$ contains a unique highest weight vector Q_{k-2j} associated to the weight $(k-2j, 0)$. Lemma 4.6.4 implies the existence of $\lambda \in \mathbb{C}$ such that

$$Q_{k-2j}(y_0, y_1, y_2, y_3) = \lambda (y_1 - i y_2)^{k-2j} (y_0^2 - y_1^2 - y_2^2 - y_3^2)^j.$$

By [10, Proposition 7.18], since $W_{\mathbb{C}}$ is a complex irreducible representation, $W_{\mathbb{C}}$ is generated by the iterated action of $\hat{\Phi}(C_2)$ and $\hat{\Phi}(C_4)$ on Q_{k-2j}, with $C_2 = R_2 + L_1 + i(R_1 + L_2)$ and $C_4 = R_2 - L_1 + i(R_1 - L_2)$. Consequently, $W_{\mathbb{C}} = \Gamma^{(k)}_{k-2j,0}$ and $W = \tilde{\Gamma}^{(k)}_{k-2j,0}$.

Case 2: There is no k such that W is included in $\mathbb{R}_k[y_0, y_1, y_2, y_3]$.

By (4.6.7) and uniqueness of this decomposition, we deduce, as previously, that there exists $j \in [\![0, [n/2]]\!]$ such that $W \simeq \Gamma_{n-2j,0}$. Considering the complexification $W_{\mathbb{C}} =$

$W + iW$ of W and extending the action of $SO(1,3)_e$ on $W_{\mathbb{C}}$ to an action of $so_{\mathbb{C}}(1,3)$, we deduce that $W_{\mathbb{C}}$ contains a highest weight vector Q_{n-2j} associated to the weight $(n-2j, 0)$. Lemma 4.6.4 implies the existence of constants $\lambda_k \in \mathbb{C}$ for $k = 0, \ldots, j$ such that

$$Q_{n-2j}(y_0, y_1, y_2, y_3) = (y_1 - iy_2)^{n-2j} \sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k}.$$

As previously, $W_{\mathbb{C}}$ is generated by the iterated action of $\hat{\Phi}(C_2)$ and $\hat{\Phi}(C_4)$ on Q_{n-2j}. Consequently, $W_{\mathbb{C}}$ is the complex vector space generated by

$$\left(\sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} \sum_{r=\max(l-m,0)}^{\min(l,n-2j-m)} \frac{(n-2j-m)!}{(n-2j-m-r)!} \frac{m!}{(m-l+r)!} \binom{l}{r} \right.$$
$$\left. (y_1 - iy_2)^{n-2j-m-r} (y_0 + y_3)^r (y_0 - y_3)^{m-l+r} (y_1 + iy_2)^{l-r} \right)_{0 \leq l, m \leq n-2j}. \quad (4.6.14)$$

For $l = 0$ and $m = n - 2j$, we deduce that

$$\sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} (y_0 - y_3)^{n-2j}$$

belongs to $W_{\mathbb{C}}$. The coefficients λ_k may not be all equal to 0. Without loss of generality, we may assume that there exists $k \in \mathbb{N}$ such that $\Re(\lambda_k)$ is non-zero. (If not, it suffices to replace λ_k with $i\lambda_k$). We obtain that

$$\sum_{k=0}^{j} \Re(\lambda_k)(y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} (y_0 - y_3)^{n-2j} \quad (4.6.15)$$

is non-zero. By definition of $W_{\mathbb{C}}$, the polynomial (4.6.15) belongs to $W_{\mathbb{C}}$. Applying $\hat{\Phi}(C_1)$ $n - 2j$ times to (4.6.15) leads to

$$\sum_{k=0}^{j} \Re(\lambda_k)(y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} (y_1 - iy_2)^{n-2j}. \quad (4.6.16)$$

But $W_{\mathbb{C}}$ is stable under $\hat{\Phi}(C_1)$, thus the polynomial (4.6.16) belongs to $W_{\mathbb{C}}$. By Lemma 4.6.4, the polynomial (4.6.16) is a highest weight vector associated to the weight $(n - 2j, 0)$. By [10, Proposition 7.15], $W_{\mathbb{C}}$ contains, up to a constant, a unique highest weight vector associated to the weight $(n - 2j, 0)$. Consequently, the coefficients λ_k of (4.6.14) are real. Finally, we deduce that W is the real vector space generated by the real parts and the imaginary parts of (4.2.2) with $\lambda_k \in \mathbb{R}$ for $k = 0, \ldots, j$. □

Appendix: Representation theory in the classical case

Thanks to the representation theory of Lie groups and Lie algebras, we can determine the finite dimensional subspaces of $\mathbb{R}[v_1, v_2, v_3]$ that are stable under any rotation of $SO(3)$. We consider the following action of $SO(3)$ on the subspace $\mathcal{P}_n$ composed of the polynomials of $\mathbb{R}[y_1, y_2, y_3]$ with total degree less or equal to n.

$$\begin{array}{rcl} \varphi : SO(3) & \longrightarrow & GL(\mathcal{P}_n) \\ L & \longmapsto & \{R(y_1, y_2, y_3) \mapsto R(L^{-1}(y_1, y_2, y_3))\} \end{array} \tag{4.6.17}$$

We first consider the restriction of φ to $\mathbb{R}_n[y_1, y_2, y_3]$, the set of real homogeneous polynomials with degree n. The representation theory of Lie groups and Lie algebras then implies that

Theorem 4.6.5 *The representation $(\varphi_{|\mathbb{R}_n[y_1,y_2,y_3]}, \mathbb{R}_n[y_1, y_2, y_3])$ of $SO(3)$ is not irreducible. We have the following decomposition:*

$$\mathbb{R}_n[y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \overline{\Gamma}_{2n-4j}^{(n)}, \tag{4.6.18}$$

where the space $\overline{\Gamma}_{2n-4j}^{(n)}$ is generated by the real parts and the imaginary parts of

$$(y_1^2 + y_2^2 + y_3^2)^j \sum_{m=[(l+1)/2]}^{\min(l,n-2j)} \frac{(-1)^{m+l}\,(n-2j)!\,l!}{(n-2j-m)!(l-m)!(2m-l)!\,2^{n-2j+l-2m}} \, y_1^{2m-l}(y_2 - iy_3)^{n-2j-m}(y_2 + iy_3)^{l-m}, \tag{4.6.19}$$

for $l \in [\![0, 2n-4j]\!]$. The subrepresentations $(\varphi_{|\overline{\Gamma}_{2n-4j}^{(n)}}, \overline{\Gamma}_{2n-4j}^{(n)})$ are irreducible.

The proof of Theorem 4.6.5 follows the same lines as the proof of Theorem 4.6.3. As we deduced Theorem 4.2.1 from Theorem 4.6.3, we deduce the following theorem from Theorem 4.6.5.

Theorem 4.6.6 *A space W is an irreducible subrepresentation of $(\varphi, \mathcal{P}_n)$ if and only if there exist $j \in [\![0, [n/2]]\!]$ and some real numbers $(\lambda_k)_{0 \le k \le j}$ such that W is generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k (y_1^2 + y_2^2 + y_3^2)^k \sum_{m=[(l+1)/2]}^{\min(l,n-2j)} \frac{(-1)^{m+l}\,(n-2j)!\,l!}{(n-2j-m)!(l-m)!(2m-l)!\,2^{n-2j+l-2m}} \, y_1^{2m-l}(y_2 - iy_3)^{n-2j-m}(y_2 + iy_3)^{l-m}, \tag{4.6.20}$$

for $l \in [\![0, 2n-4j]\!]$.

This theorem describes all the irreducible representations of $(\varphi, \mathcal{P}_n)$. We then obtain all the finite dimensional subspaces of $\mathbb{R}[v_1, v_2, v_3]$ that are stable under any rotation. We have the following proposition.

Proposition 4.6.7 *For every $r \in \mathbb{N}$, $j \in [\![0, [r/2]]\!]$, let $\mathbb{T}_{r,j}$ denote the vector space generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k |v|^{2k} \sum_{m=[(l+1)/2]}^{\min(l,r-2j)} \frac{(-1)^{m+l}\,(r-2j)!\,l!}{(r-2j-m)!(l-m)!(2m-l)!\,2^{r-2j+l-2m}} v_1^{2m-l}(v_2-iv_3)^{r-2j-m}(v_2+iv_3)^{l-m},$$

for $l \in [\![0, 2r-4j]\!]$. Each $\mathbb{T}_{r,j}$ is stable under any rotation.

Moreover, a finite dimensional subspace $\mathbb{T}$ of $\mathbb{R}[v_1, v_2, v_3]$ is stable under any rotation if and only if there exist $N \in \mathbb{N}$ and some $r_k \in \mathbb{N}$ and $j_k \in [\![0, [r_k/2]]\!]$, $k = 1, \ldots, N$ such that $\mathbb{T}$ is the vector sum of the $\mathbb{T}_{r_k,j_k}$, $k = 1, \ldots, N$.

Let us now show that, for each $r \in \mathbb{N}$, the spaces $\mathbb{T}_{r,0}$ are generated by the components of some tensors.

Theorem 4.6.8 *Let $l \in \mathbb{N}$. For any tensor T of order l, we denote by $\overline{T}$ the symmetric part of T, that is the tensor whose components are*

$$\overline{T}^{j_1,\ldots,j_l} = \frac{1}{l!} \sum_{\sigma \in \Sigma_l} T^{j_{\sigma(1)},\ldots,j_{\sigma(l)}}, \qquad (j_1, \ldots, j_l) \in [\![1, 3]\!]^l,$$

where Σ_l denotes the symmetric group of order l.

Then, the vector space $\mathbb{T}_{r,0}$ given by Proposition 4.6.7 is generated by the components of the tensor $S_r(v)$ defined by

$$S_r(v) = \mathcal{T}_r(v) + \sum_{k=1}^{[r/2]} \frac{(-1)^k\, r!\,(r-1)!\,(2r-2k)!}{2\,(r-2k)!\,k!\,(r-k)!\,(2r-1)!}\, |v|^{2k}\, \overline{\underbrace{I_3 \otimes \ldots \otimes I_3}_{k \text{ times}} \otimes \mathcal{T}_{r-2k}(v)}, \tag{4.6.21}$$

where $\mathcal{T}_r(v) = \otimes^r v$ and I_3 is the identity matrix of order 3.

We now write down the moment spaces that arise in (4.6.18) for $n = 2, 3, 4$. Since we look here for moment spaces that are compatible with the Galilean invariance, we also consider the stability under the translations. Moreover, we are only interested in moment spaces that generalize the fluid dynamic approximation and thus contain the mass 1, the velocity v and the energy $|v|^2$.

Case $n = 2$

By Theorem 4.6.5, we have

$$\mathbb{R}_2[v_1, v_2, v_3] = \overline{\Gamma}_4^{(2)} \oplus \overline{\Gamma}_0^{(2)},$$

with $\overline{\Gamma}_0^{(2)} = \text{span}\,(|v|^2)$ and

$$\overline{\Gamma}_4^{(2)} = \text{span}\,((v_i v_j)_{i \neq j},\, v_1^2 - v_3^2,\, v_2^2 - v_3^2).$$

We add the mass and the velocity to $\overline{\Gamma}_0^{(2)}$ and obtain the moment space span $(1, v, |v|^2)$.

The space $\overline{\Gamma}_4^{(2)}$ is a 5-dimensional space. Adding 1, v and $|v|^2$, we obtain the 10-dimensional space span $(1,\, v,\, v \otimes v)$ which is stable under any rotation and any translation.

Case $n = 3$
We infer from (4.6.18) that

$$\mathbb{R}_3[v_1, v_2, v_3] = \overline{\Gamma}_6^{(3)} \oplus \overline{\Gamma}_2^{(3)},$$

where $\overline{\Gamma}_2^{(3)} = \text{span}\,(v|v|^2)$ and

$$\begin{aligned} \overline{\Gamma}_6^{(3)} \;\; = \;\; & \text{span}\,(v_1 v_2 v_3,\, v_1(v_1^2 - 3v_2^2),\, v_1(v_2^2 - v_3^2),\, v_2(v_1^2 - v_3^2), \\ & v_2(v_2^2 - 3v_3^2),\, v_3(v_1^2 - v_2^2),\, v_3(3v_2^2 - v_3^2)). \end{aligned}$$

The space $\overline{\Gamma}_2^{(3)}$ is a 3-dimensional space. Adding 1, v and $|v|^2$, we obtain the 8-dimensional space span $(1,\, v,\, |v|^2,\, v|v|^2)$ which is stable under any rotation but not under the translations. In order to make it stable under any translation, we add $v \otimes v$ and obtain the Grad 13-moment system

$$\text{span}\,(1,\, v,\, v \otimes v,\, v|v|^2).$$

The space $\overline{\Gamma}_6^{(3)}$ is a 7-dimensional space. Adding 1, v and $|v|^2$, we obtain the 12-dimensional space

$$\begin{aligned} & \text{span}\,(1,\, v,\, |v|^2,\, v_1 v_2 v_3,\, v_1(v_1^2 - 3v_2^2),\, v_1(v_2^2 - v_3^2), \\ & \qquad v_2(v_1^2 - v_3^2),\, v_2(v_2^2 - 3v_3^2),\, v_3(v_1^2 - v_2^2),\, v_3(3v_2^2 - v_3^2)). \end{aligned}$$

This space is not stable under the translations. Consequently, we add $v \otimes v$ and get

$$\begin{aligned} & \text{span}\,(1,\, v,\, v \otimes v,\, v_1 v_2 v_3,\, v_1(v_1^2 - 3v_2^2),\, v_1(v_2^2 - v_3^2), \\ & \qquad v_2(v_1^2 - v_3^2),\, v_2(v_2^2 - 3v_3^2),\, v_3(v_1^2 - v_2^2),\, v_3(3v_2^2 - v_3^2)), \end{aligned}$$

which is stable under any rotation and under any translations. This space has dimension 17. We can also add $v \otimes v \otimes v$ and obtain the system

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v),$$

which has dimension 20.

Case $n = 4$
By (4.6.18), we have

$$\mathbb{R}_4[v_1, v_2, v_3] = \overline{\Gamma}_8^{(4)} \oplus \overline{\Gamma}_4^{(4)} \oplus \overline{\Gamma}_0^{(4)},$$

where $\overline{\Gamma}_0^{(4)} = \text{span}\,(|v|^4)$,

$$\overline{\Gamma}_4^{(4)} = |v|^2\,\text{span}\,((v_iv_j)_{i\neq j},\, v_1^2 - v_3^2,\, v_2^2 - v_3^2),$$

and

$$\begin{aligned}\overline{\Gamma}_8^{(4)} \quad = \quad & \text{span}\,(v_2^4 - 6v_2^2v_3^2 + v_3^4,\, 8v_1^4 - 24v_1^2(v_2^2 + v_3^2) + 3(v_2^2 + v_3^2)^2,\, v_2v_3(v_2^2 - v_3^2),\\ & v_1v_2(v_2^2 - 3v_3^2),\, v_1v_3(3v_2^2 - v_3^2),\, v_1v_2(4v_1^2 - 3v_2^2 - 3v_3^2),\\ & v_1v_3(4v_1^2 - 3v_2^2 - 3v_3^2),\, v_2v_3(6v_1^2 - v_2^2 - v_3^2),\, v_3^4 - 6v_1^2v_3^2 + 6v_1^2v_2^2 - v_2^4).\end{aligned}$$

The space $\overline{\Gamma}_0^{(4)}$ is a 1-dimensional space. Adding 1, v and $|v|^2$, we obtain the 6-dimensional space span $(1,\, v,\, |v|^2,\, |v|^4)$ which is stable under any rotation but not under the translations. In order to make it stable under any translation, we add $v \otimes v$ and $v|v|^2$. We then get

$$\text{span}\,(1,\, v,\, v \otimes v,\, v|v|^2, |v|^4),$$

which is a 14-dimensional space. We can also add $v \otimes v \otimes v$ instead of $v|v|^2$ and obtain

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^4),$$

which is a 21-dimensional space.

The space $\overline{\Gamma}_4^{(4)}$ is a 5-dimensional space. Adding 1, v and $|v|^2$, we obtain the 10-dimensional space

$$\text{span}\,(1,\, v,\, |v|^2,\, |v|^2(v_iv_j)_{i\neq j},\, |v|^2(v_1^2 - v_3^2),\, |v|^2(v_2^2 - v_3^2)).$$

This space is not stable under the translations. Consequently, we add $v \otimes v$ and $v \otimes v \otimes v$. We get

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^2(v_iv_j)_{i\neq j},\, |v|^2(v_1^2 - v_3^2),\, |v|^2(v_2^2 - v_3^2)).$$

which is stable under any rotation and any translation. This space has dimension 25. We could also add either $|v|^2 v \otimes v$ or $v \otimes v \otimes v \otimes v$ to the previous space. We would then get

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^2 v \otimes v),$$

and

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, v \otimes v \otimes v \otimes v).$$

We add 1, v and $|v|^2$ to the space $\overline{\Gamma}_8^{(4)}$ and obtain the space

$$\begin{aligned}\text{span}\,(&1, v, |v|^2, v_2^4 - 6v_2^2v_3^2 + v_3^4,\, 8v_1^4 - 24v_1^2(v_2^2 + v_3^2) + 3(v_2^2 + v_3^2)^2,\\ & v_2v_3(v_2^2 - v_3^2),\, v_1v_2(v_2^2 - 3v_3^2),\, v_1v_3(3v_2^2 - v_3^2),\, v_1v_2(4v_1^2 - 3v_2^2 - 3v_3^2),\\ & v_1v_3(4v_1^2 - 3v_2^2 - 3v_3^2),\, v_2v_3(6v_1^2 - v_2^2 - v_3^2),\, v_3^4 - 6v_1^2v_3^2 + 6v_1^2v_2^2 - v_2^4).\end{aligned}$$

But this space is not stable under the translations and we need to add $v \otimes v$ and $v \otimes v \otimes v$. We get the following 29-dimensional space

$$\begin{aligned}\text{span}\,(1, v, v\otimes v, v\otimes v\otimes v, v_2^4 - 6v_2^2v_3^2 + v_3^4,\ 8v_1^4 - 24v_1^2(v_2^2+v_3^2) + 3(v_2^2+v_3^2)^2,\\ v_2v_3(v_2^2-v_3^2),\ v_1v_2(v_2^2-3v_3^2),\ v_1v_3(3v_2^2-v_3^2),\ v_1v_2(4v_1^2-3v_2^2-3v_3^2),\\ v_1v_3(4v_1^2-3v_2^2-3v_3^2),\ v_2v_3(6v_1^2-v_2^2-v_3^2),\ v_3^4-6v_1^2v_3^2+6v_1^2v_2^2-v_2^4).\end{aligned}$$

Conclusion
The classical moment spaces with maximal degree 2, 3 or 4 are

$$\begin{array}{lll}
\text{degree} = 2 & \text{span}(1, v, |v|^2), & \text{(admissible)}\\
& \text{span}(1, v, v\otimes v), & \text{(admissible)}\\
\text{degree} = 3 & \text{span}(1, v, v\otimes v, v|v|^2), & \text{(non admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v - 3/5|v|^2\overline{I_3\otimes v}), & \text{(non admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v), & \text{(non admissible)}\\
\text{degree} = 4 & \text{span}(1, v, v\otimes v, v|v|^2, |v|^4), & \text{(admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^4), & \text{(admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^2(v\otimes v - |v|^2/3I_3)), & \text{(non admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^2 v\otimes v), & \text{(admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v & \\
& \qquad - 6|v|^2/7\overline{I_3\otimes v\otimes v} + 3|v|^4/35 I_3\otimes I_3), & \text{(non admissible)}\\
& \text{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v), & \text{(admissible)}
\end{array}$$

which have respectively dimension 5, 10, 13, 17, 20, 14, 21, 25, 26, 29 and 35.

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Annexe B

Théorie de la représentation des groupes de Lie et algèbres de Lie

B.1 Le groupe propre de Lorentz et son algèbre de Lie

B.1.1 Le groupe propre de Lorentz

On considère ici l'espace $\mathbb{R}^4$ de Minkowski. Il est muni d'une forme bilinéaire symétrique non-dégénérée g définie par

$$g(x,y) = x_0\,y_0 - x_1\,y_1 - x_2\,y_2 - x_3\,y_3, \qquad x,\,y \in \mathbb{R}^4,$$

que l'on appelle produit scalaire ou tenseur métrique.

L'ensemble des matrices réelles $L \in \mathcal{M}(4,\mathbb{R})$ qui laissent g invariant (i.e. telles que $g(Lx, Ly) = g(x,y)$ pour tous $x, y \in \mathbb{R}^4$) constitue le groupe orthogonal généralisé $O(1,3)$. C'est un sous-groupe de $GL(4,\mathbb{R})$ et un groupe de Lie matriciel. Rappelons la définition d'un groupe de Lie matriciel :

Définition B.1.1 *On appelle groupe de Lie matriciel tout sous-groupe G de $GL(n,\mathbb{C})$ tel que G est fermé dans $GL(n,\mathbb{C})$, c'est à dire tel que, si (A_m) est une suite de matrices dans G qui converge vers une matrice A, alors, soit $A \in G$, soit A n'est pas inversible.*

Toute transformation L de $O(1,3)$ vérifie $\det(L) = \pm 1$. De plus, on a soit $L_{00} \geq 1$, soit $L_{00} \leq -1$. On appelle groupe propre de Lorentz l'ensemble des matrices L de $O(1,3)$ telles que $\det(L) = 1$ et $L_{00} \geq 1$ (i.e. il n'y a pas d'inversion du temps). Il correspond à la composante connexe de l'identité dans $O(1,3)$. On le note $SO(1,3)_e$. C'est encore un groupe de Lie matriciel.

B.1.2 L'algèbre de Lie de $SO(1,3)_e$

Notons que les algèbres de Lie sont indispensables pour l'étude des groupes de Lie matriciels car elles présentent l'avantage d'être des espaces vectoriels. On dispose alors des outils de l'algèbre linéaire. Rappelons tout d'abord la définition des algèbres de Lie associées aux groupes de Lie matriciels.

Définition B.1.2 *Soit G un groupe de Lie matriciel. L'algèbre de Lie de G, notée $\mathfrak{g}$, est l'ensemble des matrices X telles que e^{tX} appartient à G pour tout réel t.*

Par conséquent, l'algèbre de Lie associée à $O(1,3)$ est constituée de l'ensemble des matrices $X \in \mathcal{M}(4,\mathbb{R})$ tel que $gX^T + Xg = 0$. Or, on a le lemme suivant.

Lemme B.1.1 *Soient G un groupe de Lie matriciel et X un élément de son algèbre de Lie. Alors, e^X appartient à la composante connexe de l'identité de G.*

Pour une preuve de ce lemme, voir [2, Proposition 2.16]. On déduit alors que l'algèbre de Lie associée à $SO(1,3)_e$ coïncide avec celle associée à $O(1,3)$. On la note $so_{\mathbb{R}}(1,3)$.

$$so_{\mathbb{R}}(1,3) = \left\{X \in M(4,\mathbb{R}); gX^T + Xg = 0\right\}.$$

Notons alors $so_{\mathbb{C}}(1,3)$ le complexifié de $so_{\mathbb{R}}(1,3)$,

$$so_{\mathbb{C}}(1,3) = \left\{X \in M(4,\mathbb{C}); gX^T + Xg = 0\right\}.$$

Posons

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

et

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Les matrices $(R_1, R_2, R_3, L_1, L_2, L_3)$ forment une base du $\mathbb{C}$-espace vectoriel $so_{\mathbb{C}}(1,3)$. On considère $D_1 = iR_3$ et $D_2 = L_3$. Les matrices D_1 et D_2 sont diagonalisables et commutent. Par conséquent, elles sont simultanément diagonalisables. Posons

$$\begin{aligned} C_1 &= R_1 + L_2 + i(R_2 + L_1), & C_2 &= R_2 + L_1 + i(R_1 + L_2), \\ C_3 &= R_1 - L_2 + i(R_2 - L_1), & C_4 &= R_2 - L_1 + i(R_1 - L_2). \end{aligned}$$

Les matrices D_i et C_i vérifient alors

$$\begin{aligned} &[D_1, D_2] = 0, && [D_1, C_1] = C_1, && [D_2, C_1] = -C_1, \\ &[D_1, C_2] = -C_2, && [D_1, C_3] = C_3, && [D_1, C_4] = -C_4, \\ &[D_2, C_2] = -C_2, && [D_2, C_3] = C_3, && [D_2, C_4] = C_4, \\ &[C_1, C_2] = 0, && [C_1, C_3] = 0, && [C_1, C_4] = 4i(D_2 - D_1), \\ &[C_2, C_3] = 4i(D_1 + D_2), && [C_2, C_4] = 0, && [C_3, C_4] = 0, \end{aligned} \tag{B.1.1}$$

où $[A,B] = AB - BA$, pour toutes matrices A, B.

Le sous-espace de $so_{\mathbb{C}}(1,3)$ engendré par D_1 et D_2 est une sous-algèbre de Cartan de $so_{\mathbb{C}}(1,3)$, notée $\mathfrak{h}$. Rappelons la définition des sous-algèbres de Cartan.

Définition B.1.3 *Soit $\mathfrak{g}$ une algèbre de Lie. On appelle sous-algèbre de Cartan de $\mathfrak{g}$ un sous-espace vectoriel $\mathfrak{h}$ de $\mathfrak{g}$ tel que*

1. *Pour tous $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$.*
2. *Pour tout $X \in \mathfrak{g}$, si $[H,X] = 0$ pour tout $H \in \mathfrak{h}$, alors $X \in \mathfrak{h}$.*
3. *Pour tout $H \in \mathfrak{h}$, l'application $ad_H : \mathfrak{g} \longrightarrow \mathfrak{g}$ définie par $ad_H(X) = [H,X]$ est diagonalisable.*

Un couple $\alpha = (a_1, a_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$ est appelé RACINE de $so_{\mathbb{C}}(1,3)$ si il existe un élément non-nul $Z \in so_{\mathbb{C}}(1,3)$ tel que

$$[D_1, Z] = a_1 Z \qquad \text{et} \qquad [D_2, Z] = a_2 Z.$$

D'après (B.1.1), les racines de $so_{\mathbb{C}}(1,3)$ sont

$$\alpha_1 = (1,-1), \qquad \alpha_2 = (-1,-1), \qquad \alpha_3 = (1,1) \qquad \text{et} \qquad \alpha_4 = (-1,1). \tag{B.1.2}$$

On a

$$so_{\mathbb{C}}(1,3) = \mathfrak{h} \oplus \bigoplus_{j \in \{1,\ldots,4\}} g_{\alpha_j},$$

où, pour chaque $j \in \{1, \ldots, 4\}$, g_{α_j} désigne l'espace vectoriel engendré par C_j.

B.2 Représentations irréductibles de dimension finie de l'algèbre de Lie $so_{\mathbb{C}}(1,3)$

B.2.1 Quelques rappels sur les représentations

Définitions

Définition B.2.1 *On appelle représentation d'un groupe de Lie G sur un espace vectoriel V un homomorphisme de groupes de Lie*

$$\Pi : G \longrightarrow GL(V),$$

c'est à dire, une application continue $\Pi : G \longrightarrow GL(V)$ telle que

$$\Pi(gh) = \Pi(g)\Pi(h), \qquad g, h \in G.$$

On appelle représentation d'une algèbre de Lie $\mathfrak{g}$ sur un espace vectoriel V un homomorphisme d'algèbres de Lie

$$\pi : \mathfrak{g} \longrightarrow gl(V) = \mathcal{L}(V),$$

c'est à dire, une application linéaire $\pi : \mathfrak{g} \longrightarrow \mathcal{L}(V)$ *telle que*

$$\pi([X,Y]) = [\pi(X), \pi(Y)], \qquad X, Y \in \mathfrak{g},$$

où $\mathcal{L}(V)$ *est l'ensemble des endomorphismes de* V *et constitue l'algèbre de Lie de* $GL(V)$.

Définition B.2.2 *Soit* (Π, V) *une représentation d'un groupe* G. *Un sous-espace* W *de* V *est dit stable sous l'action de* G *si* $\Pi(g)w \in W$ *pour tout* $w \in W$ *et tout* $g \in G$.

La représentation (Π, V) *est dite irréductible s'il n'y a pas de sous-espace stable non trivial, c'est à dire différent du* $\{0\}$ *ou de* V.

On définit de la même manière les notions de sous-espaces stables et de représentations irréductibles pour une algèbre de Lie.

Exemples de représentations

La représentation triviale Considérons un espace vectoriel V de dimension finie. Etant donné un groupe de Lie matriciel G, on définit la représentation triviale de G, $\Pi : G \longrightarrow GL(V)$ par la formule

$$\Pi(g) = Id, \qquad g \in G.$$

Si $\mathfrak{g}$ est une algèbre de Lie, on définit la représentation triviale de $\mathfrak{g}$, $\pi : \mathfrak{g} \longrightarrow \mathcal{L}(V)$ par

$$\pi(X) = 0, \qquad X \in \mathfrak{g}.$$

La représentation standard Un groupe de Lie matriciel G est, par définition, un sous-ensemble de $GL(n, \mathbb{C})$. L'inclusion de G dans $GL(n, \mathbb{C})$ (i.e. $\Pi(A) = A$) est une représentation de G appelée représentation standard de G. Par exemple, la représentation standard de $SO(3)$ est l'action usuelle de $SO(3)$ sur $\mathbb{R}^3$.

Si G est un sous-groupe de $GL(n, \mathbb{C})$ alors son algèbre de Lie $\mathfrak{g}$ est une sous-algèbre de $gl(n, \mathbb{C}) = \mathcal{M}(n, \mathbb{C})$. L'inclusion de $\mathfrak{g}$ dans $gl(n, \mathbb{C})$ est une représentation de $\mathfrak{g}$, appelée représentation standard.

La représentation adjointe Soit G un groupe de Lie matriciel d'algèbre de Lie $\mathfrak{g}$. La représentation adjointe de G, $Ad : G \longrightarrow GL(\mathfrak{g})$ est donnée par

$$Ad_A(X) = AXA^{-1}, \qquad A \in G, X \in \mathfrak{g}.$$

Si $\mathfrak{g}$ est une algèbre de Lie, la représentation adjointe de $\mathfrak{g}$, $ad : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$ est définie par

$$ad_X(Y) = [X, Y], \qquad X, Y \in \mathfrak{g}.$$

B.2.2 La représentation adjointe de $so_{\mathbb{C}}(1,3)$

Considérons la représentation adjointe de $so_{\mathbb{C}}(1,3)$, c'est à dire l'action de $so_{\mathbb{C}}(1,3)$ sur lui-même par

$$\begin{array}{rcl} ad : so_{\mathbb{C}}(1,3) & \longrightarrow & gl(so_{\mathbb{C}}(1,3)) \\ X & \longmapsto & ad_X(Y) = [X,Y] \end{array}$$

La sous-algèbre de Cartan $\mathfrak{h}$ agit sur chaque C_k par multiplication par un scalaire, i.e. $ad_{D_j}(C_k) = [D_j, C_k] = \alpha_k(D_j)C_k$. En ce qui concerne l'action de C_k sur C_l, on a

$$\begin{array}{rcl} [D_j,[C_k,C_l]] & = & [C_k,[D_j,C_l]] + [[D_j,C_k],C_l] \\ & = & (\alpha_l(D_j) + \alpha_k(D_j))[C_k,C_l], \end{array}$$

et donc,

$$ad_{C_k}(g_{\alpha_l}) \subset g_{\alpha_l+\alpha_k}.$$

B.2.3 Les représentations irréductibles de $so_{\mathbb{C}}(1,3)$

Soit (π, V) une représentation de dimension finie de $so_{\mathbb{C}}(1,3)$. On appelle POIDS de cette représentation tout couple $\mu = (m_1, m_2) \in \mathbb{C}^2$ tel qu'il existe un vecteur non-nul $v \in V$ vérifiant

$$\pi(D_1)v = m_1 v \qquad \text{et} \qquad \pi(D_2)v = m_2 v. \tag{B.2.1}$$

On appelle VECTEUR POIDS tout vecteur v non-nul vérifiant (B.2.1). De plus, si $\mu = (m_1, m_2)$ est un poids, l'espace vectoriel engendré par tous les vecteurs v vérifiant (B.2.1) est appelé espace de poids associé au poids μ. On le note V_μ. On a alors le résultat suivant :

Proposition B.2.3 *Soient $\mathfrak{g}$ une algèbre de Lie complexe et $\mathfrak{h}$ une algèbre de Cartan de $\mathfrak{g}$. Toute représentation irréductible complexe de dimension finie (π, V) de $\mathfrak{g}$ est la somme directe de ses espaces de poids, c'est à dire que l'ensemble des opérateurs de la forme $\pi(H)$, $H \in \mathfrak{h}$, sont simultanément diagonalisables dans toute représentation irréductible complexe de dimension finie.*

Pour une preuve de cette proposition, on réfère à [2, Theorem 7.12]. Etant donnée une représentation irréductible complexe de dimension finie (π, V), on a, par conséquent, $V = \oplus V_\mu$, décomposition de V en somme d'espaces propres sous l'action de $\{D_1, D_2\}$.

Chaque espace g_α agit en envoyant un espace V_μ sur un autre. En effet, si $X \in g_\alpha$ et $v \in V_\mu$, on a

$$\begin{array}{rcl} \pi(D_j)\big(\pi(X)v\big) & = & \pi(X)\big(\pi(D_j)v\big) + [\pi(D_j), \pi(X)]v \\ & = & \pi(X)\big(\pi(D_j)v\big) + \pi([D_j, X])v \\ & = & (\mu(D_j) + \alpha(D_j))\pi(X)v, \end{array}$$

c'est à dire que $\pi(X)v$ est un vecteur poids associé au poids $\alpha + \mu$. On déduit que $\pi(X)(V_\mu) \subset V_{\alpha+\mu}$.

Appelons racines positives les racines α_1 et α_3 de (B.1.2). Soient μ_1 et μ_2 deux poids de la représentation (π, V). On dit que μ_1 est plus grand que μ_2 si

$$\mu_1 - \mu_2 = a\alpha_1 + b\alpha_3 \qquad \text{avec} \qquad a \geq 0 \quad \text{et} \quad b \geq 0.$$

On le note $\mu_1 \succ \mu_2$. On appelle alors PLUS HAUT POIDS de la représentation (π, V) un poids μ_0 de (π, V) tel que, pour tout poids μ de (π, V), $\mu_0 \succ \mu$.

Théorème B.2.4 (Théorème du plus haut poids) *Soit $\mathfrak{g}$ une algèbre de Lie complexe. Alors,*

1. *Toute représentation irréductible complexe de dimension finie de $\mathfrak{g}$ a un unique plus haut poids.*
2. *Deux représentations irréductibles complexes de dimension finie de $\mathfrak{g}$ sont équivalentes si et seulement si elles ont le même plus haut poids.*

Pour une preuve de ce théorème, voir [2, Proposition 7.15]. Soit (π, V) une représentation irréductible complexe de dimension finie de $so_{\mathbb{C}}(1,3)$. Soient μ_0 son plus haut poids et v un vecteur poids associé à μ_0. Le vecteur v est appelé VECTEUR DE PLUS HAUT POIDS et on a $\pi(C_1)v = 0$, $\pi(C_3)v = 0$. L'espace V est, en fait, engendré par les images de v par C_2 et C_4. De plus, on a $\dim(V_{\mu_0}) = 1$ (le vecteur v est unique, à un scalaire près). Pour une preuve de ces résultats, voir [2, Propositions 7.17 et 7.18].

Proposition B.2.5 *Si (π, V) est une représentation complexe de dimension finie de $so_{\mathbb{C}}(1,3)$ et $v \in V$ un vecteur de plus haut poids, alors la sous-représentation W de V engendrée par les images de v par applications successives de C_2 et C_4 est irréductible.*

De plus, toute représentation irréductible de $so_{\mathbb{C}}(1,3)$ est de ce type là.

Cette proposition est en fait valable pour toutes les algèbre de Lie complexes semi-simples. Pour une démonstration, voir [1, Proposition 14.13 et Observation 14.16] ou [2, Propositions 7.18 et 7.19].

Théorème B.2.6 *Si (π, V) est une représentation irréductible complexe de dimension finie de $so_{\mathbb{C}}(1,3)$, alors son plus haut poids $\mu_0 = (m_1, m_2)$ vérifie*

$$(m_1, m_2) \in \mathbb{N} \times \mathbb{Z} \qquad \text{et} \qquad |m_2| \leq m_1. \tag{B.2.2}$$

Réciproquement, pour tout couple $(m_1, m_2) \in \mathbb{N} \times \mathbb{Z}$ tel que $|m_2| \leq m_1$, il existe une unique, à une équivalence près, représentation irréductible de $so_{\mathbb{C}}(1,3)$ de plus haut poids (m_1, m_2). On la note Γ_{m_1,m_2}.

Preuve. La démonstration est similaire à celle concernant l'algèbre de Lie $sl_{\mathbb{C}}(3)$ du groupe $SL(3,\mathbb{C})$ (cf. [2, Theorem 5.9] ou [1, Lectures 12 and 13]).

Soit (π, V) une représentation irréductible de $so_{\mathbb{C}}(1,3)$. Notons $\mu = (m_1, m_2)$ son plus haut poids. Par définition, il existe un vecteur v non-nul tel que

$$\pi(D_1)v = m_1 v, \qquad \pi(D_2)v = m_2 v, \qquad \pi(C_1)v = 0 \qquad \text{et} \qquad \pi(C_3)v = 0.$$

On sait que (cf. [2, Theorem 4.12])

Lemme B.2.1 *Une base de l'algèbre de Lie $sl_{\mathbb{C}}(2)$ du groupe $SL(2,\mathbb{C})$ est donnée par*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

et on a

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H.$$

Pour toute représentation complexe (π, V) de dimension finie de $sl_{\mathbb{C}}(2)$, si v est un vecteur non-nul de V tel que

$$\pi(X)v = 0 \qquad et \qquad \pi(H)v = \lambda v,$$

alors $\lambda \in \mathbb{N}$.

Comme $\{D_1 - D_2,\, e^{i\pi/4}C_1/2,\, e^{i\pi/4}C_4/2\}$ engendre une sous-algèbre de $so_{\mathbb{C}}(1,3)$ isomorphe à l'algèbre de Lie $sl_{\mathbb{C}}(2)$, comme $\pi(D_1 - D_2)v = (m_1 - m_2)v$ et $\pi(C_1)v = 0$, on déduit du Lemme B.2.1 que $m_1 - m_2 \in \mathbb{N}$.

De même, $\{D_1 + D_2,\, e^{i\pi/4}C_3/2,\, e^{i\pi/4}C_2/2\}$ engendre une sous-algèbre de $so_{\mathbb{C}}(1,3)$ isomorphe à $sl(2,\mathbb{C})$, $\pi(D_1 + D_2)v = (m_1 + m_2)v$ et $\pi(C_3)v = 0$. Par conséquent, on déduit du Lemme B.2.1 que $m_1 + m_2 \in \mathbb{N}$. Ainsi, (m_1, m_2) vérifie (B.2.2).

Réciproquement, soit $(m_1, m_2) \in \mathbb{N} \times \mathbb{Z}$ tel que $|m_2| \leq m_1$. Si $(m_1, m_2) = (0,0)$, la représentation triviale convient. Si $(m_1, m_2) = (1,0)$, la représentation standard convient.

Considérons le cas où $(m_1, m_2) = (1,1)$ ou $(m_1, m_2) = (1,-1)$. On pose $V = \mathbb{C}^4$ et on note π l'action standard de $so_{\mathbb{C}}(1,3)$ sur V. Soit $\Lambda^2 V$ la puissance extérieure de l'espace vectoriel V (cf. [1, Appendix B.2]). On définit

$$\begin{aligned} \tilde{\pi} : so_{\mathbb{C}}(1,3) &\longrightarrow gl(\Lambda^2 V) \\ X &\longmapsto \tilde{\pi}(X) : x_1 \wedge x_2 \mapsto \pi(X)x_1 \wedge x_2 + x_1 \wedge \pi(X)x_2. \end{aligned} \tag{B.2.3}$$

Soit (e_1, e_2, e_3, e_4) une base de V telle que, dans la représentation standard, e_1, e_2, e_3 et e_4 sont respectivement des vecteurs de poids $(1,0)$, $(0,1)$, $(-1,0)$ et $(0,-1)$. On vérifie alors facilement que $e_1 \wedge e_2$ et $e_1 \wedge e_4$ sont des vecteurs de plus haut poids de $(\tilde{\pi}, \Lambda^2 V)$ associés respectivement à $(1,1)$ et $(1,-1)$.

Traitons maintenant le cas général. Soit $(m_1, m_2) \in \mathbb{N} \times \mathbb{Z}$ tel que $|m_2| \leq m_1$. Soit (π_1, V_1) la représentation standard et v_1 un vecteur de V_1 de plus haut poids $(1,0)$. On désigne par (π_2, V_2) la représentation définie par (B.2.3). Soient v_2 et v_2' des vecteurs de V_2 de plus haut poids $(1,1)$ et $(1,-1)$ respectivement. Si $m_2 \geq 0$, on pose

$$W = \underbrace{V_1 \otimes \ldots \otimes V_1}_{m_1 - m_2 \text{ fois}} \otimes \underbrace{V_2 \otimes \ldots \otimes V_2}_{m_2 \text{ fois}}.$$

Alors, le vecteur

$$w_{m_1 - m_2, m_2} = \underbrace{v_1 \otimes \ldots \otimes v_1}_{m_1 - m_2 \text{ fois}} \otimes \underbrace{v_2 \otimes \ldots \otimes v_2}_{m_2 \text{ fois}}$$

est un vecteur de plus haut poids (m_1, m_2). Par conséquent, le plus petit sous-espace de W contenant $w_{m_1 - m_2, m_2}$ et invariant par $so_{\mathbb{C}}(1,3)$ est la représentation irréductible Γ_{m_1, m_2}.

Si au contraire, $m_2 \leq 0$, on considère l'espace

$$W = \underbrace{V_1 \otimes \ldots \otimes V_1}_{m_1+m_2 \text{ fois}} \otimes \underbrace{V_2 \otimes \ldots \otimes V_2}_{|m_2| \text{ fois}}$$

et le vecteur

$$w_{m_1+m_2,|m_2|} = \underbrace{v_1 \otimes \ldots \otimes v_1}_{m_1+m_2 \text{ fois}} \otimes \underbrace{v_2' \otimes \ldots \otimes v_2'}_{|m_2| \text{ fois}}.$$

Comme $w_{m_1+m_2,|m_2|}$ est un vecteur de plus haut poids (m_1, m_2), on conclut comme précédemment. □

B.3 Preuve du Théorème 4.6.1

Rappelons tout d'abord les notations de la Section 4.6 et l'énoncé du théorème. Notons $\mathbb{C}_n[y_0, y_1, y_2, y_3]$ l'ensemble des polynômes complexes homogènes de degré n. On considère la représentation suivante de $SO(1,3)_e$:

$$\begin{array}{rcl} \tilde{\varphi} : SO(1,3)_e & \longrightarrow & GL(\mathbb{C}_n[y_0, y_1, y_2, y_3]) \\ L & \longmapsto & \{R(y_0, y_1, y_2, y_3) \longmapsto R(L^{-1}(y_0, y_1, y_2, y_3))\} \end{array}$$

D'après [2, Proposition 4.4], la représentation $(\tilde{\varphi}, \mathbb{C}_n[y_0, y_1, y_2, y_3])$ de $SO(1,3)_e$ induit une unique représentation $(\Phi, \mathbb{C}_n[y_0, y_1, y_2, y_3])$ de $so_{\mathbb{R}}(1,3)$. Elle est définie par

$$\Phi(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ})\Big|_{t=0}, \qquad Z \in so_{\mathbb{R}}(1,3).$$

Selon [2, Proposition 4.6], cette représentation complexe de dimension finie de $so_{\mathbb{R}}(1,3)$ s'étend de manière unique en une représentation complexe $(\tilde{\Phi}, \mathbb{C}_n[y_0, y_1, y_2, y_3])$ de $so_{\mathbb{C}}(1,3)$. On a

$$\tilde{\Phi}(Z) = \Phi(Z_1) + i\Phi(Z_2), \qquad Z = Z_1 + iZ_2 \in so_{\mathbb{C}}(1,3), \quad Z_1, Z_2 \in so_{\mathbb{R}}(1,3).$$

Théorème B.3.1 *La représentation $(\tilde{\Phi}, \mathbb{C}_n[y_0, y_1, y_2, y_3])$ de $so_{\mathbb{C}}(1,3)$ n'est pas irréductible. Elle se décompose en somme de représentations irréductibles de la manière suivante :*

$$\mathbb{C}_n[y_0, y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \Gamma^{(n)}_{n-2j,0},$$

où $\Gamma^{(n)}_{n-2j,0}$ est une représentation irréductible de poids $(n-2j, 0)$ isomorphe à la représentation $\Gamma_{n-2j,0}$ donnée par le Théorème B.2.6. Une base de $\Gamma^{(n)}_{n-2j,0}$ est donnée par

$$\left((y_0^2 - y_1^2 - y_2^2 - y_3^2)^j \sum_{m=\max(l-k,0)}^{\min(l,n-2j-k)} \frac{(n-2j-k)!}{(n-2j-k-m)!} \frac{k!}{(k-l+m)!} \binom{l}{m} \right.$$
$$\left. (y_1 - iy_2)^{n-2j-k-m} (y_0 + y_3)^m (y_0 - y_3)^{k-l+m} (y_1 + iy_2)^{l-m} \right)_{0 \leq k, l \leq n-2j} . \quad \text{(B.3.1)}$$

Preuve. Effectuons le changement de variables $Y = PX$ où $Y = (y_j)_{0\leq j\leq 3}$, $X = (x_j)_{0\leq j\leq 3}$ et

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{B.3.2}$$

La matrice de passage P permet de diagonaliser D_1 et D_2. On a

$$\tilde{D}_1 = P^{-1}D_1P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{D}_2 = P^{-1}D_2P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\tilde{C}_1 = P^{-1}C_1P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \end{pmatrix}, \qquad \tilde{C}_2 = P^{-1}C_2P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$\tilde{C}_3 = P^{-1}C_3P = \begin{pmatrix} 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{C}_4 = P^{-1}C_4P = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Par conséquent, les images de x_0, x_1, x_2 et x_3 sous l'action de $\tilde{\Phi}(\tilde{D}_1)$, $\tilde{\Phi}(\tilde{D}_2)$, $\tilde{\Phi}(\tilde{C}_1)$, $\tilde{\Phi}(\tilde{C}_2)$, $\tilde{\Phi}(\tilde{C}_3)$ et $\tilde{\Phi}(\tilde{C}_4)$ sur $\mathbb{C}_1[x_0, x_1, x_2, x_3]$ sont les suivantes :

	$\tilde{\Phi}(\tilde{D}_1)$	$\tilde{\Phi}(\tilde{D}_2)$	$\tilde{\Phi}(\tilde{C}_1)$	$\tilde{\Phi}(\tilde{C}_2)$	$\tilde{\Phi}(\tilde{C}_3)$	$\tilde{\Phi}(\tilde{C}_4)$
x_0	0	$-x_0$	0	0	$2ix_1$	$2x_2$
x_1	x_1	0	0	$-2x_0$	0	$2x_3$
x_2	$-x_2$	0	$-2ix_0$	0	$2ix_3$	0
x_3	0	x_3	$-2ix_1$	$-2x_2$	0	0

On déduit alors que

$$\tilde{\Phi}(\tilde{D}_1)(x_1^n) = nx_1^n, \qquad \tilde{\Phi}(\tilde{D}_2)(x_1^n) = 0, \qquad \tilde{\Phi}(\tilde{C}_1)(x_1^n) = 0 \qquad \text{et} \qquad \tilde{\Phi}(\tilde{C}_3)(x_1^n) = 0.$$

Le vecteur x_1^n est un vecteur de plus haut poids associé au poids $(n, 0)$. Par conséquent, $(\tilde{\Phi}, \mathbb{C}_n[x_0, x_1, x_2, x_3])$ admet une représentation irréductible isomorphe à $\Gamma_{n,0}$ comme sous-représentation. On la note $\Gamma_{n,0}^{(n)}$. Une base de cette représentation est obtenue en appliquant $\tilde{\Phi}(\tilde{C}_2)$ et $\tilde{\Phi}(\tilde{C}_4)$ au vecteur de plus haut poids associé au poids $(n, 0)$, c'est à dire x_1^n. En appliquant k fois $\tilde{\Phi}(\tilde{C}_4)$ à x_1^n, on obtient, à une constante près, $x_3^k x_1^{n-k}$, pour $0 \leq k \leq n$. De plus, on a

$$\left(\frac{-1}{2}\right)^l \tilde{\Phi}(\tilde{C}_2)^l(x_1^{n-k}x_3^k) = \sum_{j=\max(l-k,0)}^{\min(l,n-k)} \frac{(n-k)!}{(n-k-j)!}\frac{k!}{(k-l+j)!}\binom{l}{j} x_1^{n-k-j}x_0^j x_3^{k-l+j}x_2^{l-j}.$$

Par conséquent,

$$\left(\sum_{j=\max(l-k,0)}^{\min(l,n-k)} \frac{(n-k)!}{(n-k-j)!}\frac{k!}{(k-l+j)!}\binom{l}{j} x_1^{n-k-j}x_0^j x_3^{k-l+j}x_2^{l-j}\right)_{0\leq k,\, l\leq n} \tag{B.3.3}$$

forme une base de $\Gamma^{(n)}_{n,0}$.

Montrons maintenant que

$$\mathbb{C}_n[x_0,x_1,x_2,x_3] \simeq \Gamma^{(n)}_{n,0} \oplus \mathbb{C}_{n-2}[x_0,x_1,x_2,x_3]. \tag{B.3.4}$$

Pour cela, on considère l'application $\theta : \mathbb{C}_n[x_0,x_1,x_2,x_3] \longrightarrow \mathbb{C}_{n-2}[x_0,x_1,x_2,x_3]$ définie par

$$\theta\left(x_0^{k_0}x_1^{k_1}x_2^{k_2}x_3^{k_3}\right) = k_0k_3x_0^{k_0-1}x_1^{k_1}x_2^{k_2}x_3^{k_3-1} - k_1k_2x_0^{k_0}x_1^{k_1-1}x_2^{k_2-1}x_3^{k_3}.$$

L'application θ est surjective. Il suffit donc de montrer que $\ker\theta = \Gamma^{(n)}_{n,0}$. Comme

$$\dim\, \mathbb{C}_k[x_0,x_1,x_2,x_3] = \frac{1}{6}(k+1)(k+2)(k+3), \qquad k\in\mathbb{N},$$

on déduit que $\dim(\ker\theta) = (n+1)^2$. Comme (B.3.3) est une base de la représentation irréductible $\Gamma^{(n)}_{n,0}$, on déduit facilement que $\Gamma^{(n)}_{n,0}$ est inclus dans $\ker\theta$. Par égalité des dimensions, on obtient $\ker\theta = \Gamma^{(n)}_{n,0}$, ce qui termine la preuve de (B.3.4).

Par récurrence immédiate, on obtient

$$\mathbb{C}_n[x_0,x_1,x_2,x_3] = \bigoplus_{j=0}^{[n/2]} \Gamma^{(n)}_{n-2j,0}.$$

Cherchons une base de chaque représentation irréductible $\Gamma^{(n)}_{n-2j,0}$ de $(\tilde{\Phi}, \mathbb{C}_n[x_0,x_1,x_2,x_3])$, pour $0\leq j\leq [n/2]$. On vérifie facilement que, pour chaque $0\leq j\leq [n/2]$, $x_1^{n-2j}(x_3x_0 - x_1x_2)^j$ est un vecteur de plus haut poids associé à $(n-2j,0)$. De plus, on a

$$\begin{aligned}\tilde{\Phi}(\tilde{C}_4)(x_1^{n-2j}(x_3x_0-x_1x_2)^j) &= (x_3x_0-x_1x_2)^j\,\tilde{\Phi}(\tilde{C}_4)(x_1^{n-2j}) \\ &\quad + jx_1^{n-2j}\,(x_3x_0-x_1x_2)^{j-1}\,\tilde{\Phi}(\tilde{C}_4)(x_3x_0-x_1x_2).\end{aligned}$$

Or, $\tilde{\Phi}(\tilde{C}_4)(x_3x_0-x_1x_2) = 0$. Donc,

$$\tilde{\Phi}(\tilde{C}_4)(x_1^{n-2j}(x_3x_0-x_1x_2)^j) = (x_3x_0-x_1x_2)^j\,\tilde{\Phi}(\tilde{C}_4)(x_1^{n-2j}),$$

et, par récurrence immédiate,

$$\tilde{\Phi}(\tilde{C}_4)^k(x_1^{n-2j}(x_3x_0-x_1x_2)^j) = (x_3x_0-x_1x_2)^j\,\tilde{\Phi}(\tilde{C}_4)^k(x_1^{n-2j}),$$

pour $0\leq k\leq n-2j$. De même, on déduit de $\tilde{\Phi}(\tilde{C}_2)(x_3x_0-x_1x_2) = 0$ que

$$\tilde{\Phi}(\tilde{C}_2)^l\,\tilde{\Phi}(\tilde{C}_4)^k(x_1^{n-2j}(x_3x_0-x_1x_2)^j) = (x_3x_0-x_1x_2)^j\,\tilde{\Phi}(\tilde{C}_2)^l\,\tilde{\Phi}(\tilde{C}_4)^k(x_1^{n-2j}),$$

pour $0 \leq k,\, l \leq n-2j$. Ainsi,

$$\left((x_3x_0 - x_1x_2)^j \sum_{m=\max(l-k,0)}^{\min(l,n-2j-k)} \frac{(n-2j-k)!}{(n-2j-k-m)!} \frac{k!}{(k-l+m)!} \binom{l}{m} x_1^{n-2j-k-m} x_0^m x_3^{k-l+m} x_2^{l-m} \right)_{0\leq k,\, l\leq n-2j}$$

forme une base de la représentation irréductible $\Gamma^{(n)}_{n-2j,0}$ de $(\tilde{\Phi}, \mathbb{C}_n[x_0,x_1,x_2,x_3])$ pour $0 \leq j \leq [n/2]$.

Il reste maintenant à revenir aux variables $(y_k)_{0\leq k\leq 3}$. On a

$$2x_0 = y_0 + y_3, \qquad 2x_1 = y_1 - iy_2, \qquad 2x_2 = y_1 + iy_2 \qquad \text{et} \qquad 2x_3 = y_0 - y_3.$$

Ainsi, (B.3.1) forme une base de la représentation irréductible $\Gamma^{(n)}_{n-2j,0}$ de $(\tilde{\Phi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ pour $0 \leq j \leq [n/2]$. □

B.4 Représentations du groupe des rotations $SO(3)$

B.4.1 L'algèbre de Lie de $SO(3)$

L'algèbre de Lie associée à $O(3)$ est constituée de l'ensemble des matrices $X \in \mathcal{M}(3,\mathbb{R})$ tel que $X^T + X = 0$. On déduit alors du Lemme B.1.1 que l'algèbre de Lie associée à $SO(3)$ coïncide avec celle associée à $O(3)$. On la note $so_{\mathbb{R}}(3)$.

$$so_{\mathbb{R}}(3) = \left\{ X \in M(3,\mathbb{R}); X^T + X = 0 \right\}.$$

Notons $so_{\mathbb{C}}(3)$ le complexifié de $so_{\mathbb{R}}(3)$,

$$so_{\mathbb{C}}(3) = \left\{ X \in M(3,\mathbb{C}); X^T + X = 0 \right\}.$$

Le $\mathbb{C}$-espace vectoriel $so_{\mathbb{C}}(3)$ est engendré par les matrices

$$\tilde{R}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \tilde{R}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{et} \quad \tilde{R}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On considère alors $D = 2i\tilde{R}_1$, $E = i\tilde{R}_2 - \tilde{R}_3$ et $F = i\tilde{R}_2 + \tilde{R}_3$. On a

$$[D,E] = 2E, \qquad [D,F] = -2F \quad \text{et} \quad [E,F] = D.$$

Par conséquent, l'algèbre $so_{\mathbb{C}}(3)$ est isomorphe à $sl_{\mathbb{C}}(2)$.

B.4.2 Preuve du Théorème 4.6.5

De l'isomorphisme entre $so_{\mathbb{C}}(3)$ et $sl_{\mathbb{C}}(2)$, on déduit le théorème suivant

Théorème B.4.1 *Si (π, V) est une représentation irréductible complexe de dimension finie de $so_{\mathbb{C}}(3)$, alors son plus haut poids m vérifie $m \in \mathbb{N}$.*

Réciproquement, pour tout $m \in \mathbb{N}$, il existe une unique, à une équivalence près, représentation irréductible de $so_{\mathbb{C}}(3)$ de plus haut poids m. On la note Γ_m.

Notons $\mathbb{C}_n[y_1, y_2, y_3]$ l'ensemble des polynômes complexes homogènes de degré n. On considère la représentation suivante de $SO(3)$:

$$\begin{array}{rcl} \tilde{\varphi} : SO(3) & \longrightarrow & GL(\mathbb{C}_n[y_1, y_2, y_3]) \\ R & \longmapsto & \{P(y_1, y_2, y_3) \longmapsto P(R^{-1}(y_1, y_2, y_3))\} \end{array}$$

D'après [2, Proposition 4.4], la représentation $(\tilde{\varphi}, \mathbb{C}_n[y_1, y_2, y_3])$ de $SO(3)$ induit une unique représentation $(\Phi, \mathbb{C}_n[y_1, y_2, y_3])$ de $so_{\mathbb{R}}(3)$ qui est définie par

$$\Phi(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ})\Big|_{t=0}, \qquad Z \in so_{\mathbb{R}}(3).$$

Selon [2, Proposition 4.6], cette représentation complexe de dimension finie de $so_{\mathbb{R}}(3)$ s'étend de manière unique en une représentation complexe $(\tilde{\Phi}, \mathbb{C}_n[y_1, y_2, y_3])$ de $so_{\mathbb{C}}(3)$. On a

$$\tilde{\Phi}(Z) = \Phi(Z_1) + i\Phi(Z_2), \qquad Z = Z_1 + iZ_2 \in so_{\mathbb{C}}(3), \quad Z_1, Z_2 \in so_{\mathbb{R}}(3).$$

Théorème B.4.2 *La représentation $(\tilde{\Phi}, \mathbb{C}_n[y_1, y_2, y_3])$ de $so_{\mathbb{C}}(3)$ n'est pas irréductible. Elle se décompose en somme de représentations irréductibles de la manière suivante :*

$$\mathbb{C}_n[y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \Gamma_{2n-4j}^{(n)},$$

où $\Gamma_{2n-4j}^{(n)}$ est une représentation irréductible de poids $2n-4j$. Une base de $\Gamma_{2n-4j}^{(n)}$ est donnée par

$$\left((y_1^2 + y_2^2 + y_3^2)^j \sum_{m=[(l+1)/2]}^{\min(l, n-2j)} \frac{(-1)^{m+l}(n-2j)!\, l!}{2^{n-2j+l-2m}(n-2j-m)!(l-m)!(2m-l)!} \right.$$
$$\left. y_1^{2m-l}(y_2 + iy_3)^{l-m}(y_2 - iy_3)^{n-2j-m} \right)_{0 \le l \le 2n-4j}. \quad \text{(B.4.1)}$$

Preuve. Effectuons le changement de variables $Y = PX$ où $Y = (y_j)_{1 \le j \le 3}$, $X = (x_j)_{1 \le j \le 3}$ et

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix}.$$

La matrice de passage P permet de diagonaliser D. Posons

$$\tilde{D} = P^{-1}DP, \qquad \tilde{E} = P^{-1}EP \qquad \text{et} \qquad \tilde{F} = P^{-1}FP.$$

On a

$$\tilde{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{et} \qquad \tilde{F} = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Par conséquent, les images de x_1, x_2 et x_3 sous l'action de $\tilde{\Phi}(\tilde{D})$, $\tilde{\Phi}(\tilde{E})$ et $\tilde{\Phi}(\tilde{F})$, sur $\mathbb{C}_1[x_1, x_2, x_3]$ sont les suivantes :

	$\tilde{\Phi}(\tilde{D})$	$\tilde{\Phi}(\tilde{E})$	$\tilde{\Phi}(\tilde{F})$
x_1	0	$2x_2$	$-2x_3$
x_2	$2x_2$	0	x_1
x_3	$-2x_3$	$-x_1$	0

On déduit alors que

$$\tilde{\Phi}(\tilde{D})(x_2^n) = 2nx_2^n \qquad \text{et} \qquad \tilde{\Phi}(\tilde{E})(x_2^n) = 0.$$

Dans $\mathbb{C}_n([x_1, x_2, x_3])$, le vecteur x_2^n est un vecteur de plus haut poids associé au poids $2n$. La représentation engendrée par les $(\tilde{\Phi}(\tilde{F})^k(x_2^n))_{k=0,\ldots,2n}$ est une représentation irréductible associée au poids $2n$. On la note $\Gamma_{2n}^{(n)}$. C'est un espace vectoriel de dimension $2n+1$. Par récurrence, on montre que, pour $0 \leq l \leq 2n$,

$$\tilde{\Phi}(\tilde{F})^l(x_2^n) = \sum_{m=[(l+1)/2]}^{\min(l,n)} \frac{(-1)^{m+l}\, n!\, 2^{l-m}}{(n-m)!}\, \alpha_{m,l}\, x_1^{2m-l}\, x_2^{n-m}\, x_3^{l-m},$$

où

$$\alpha_{m,l+1} = (2m-l)\,\alpha_{m,l} + \alpha_{m-1,l}, \qquad \text{pour} \qquad [(l+2)/2] \leq m \leq \min(l,n), \tag{B.4.2}$$

avec

$$\begin{aligned} \alpha_{m,l} &= 0, \qquad \text{pour} \qquad m < [(l+1)/2] \quad \text{ou} \quad m > \min(l,n), \\ \alpha_{l,l} &= 1, \qquad \text{pour} \qquad l \leq n. \end{aligned}$$

Montrons maintenant que

$$\mathbb{C}_n[x_1, x_2, x_3] \simeq \Gamma_{2n}^{(n)} \oplus \mathbb{C}_{n-2}[x_1, x_2, x_3]. \tag{B.4.3}$$

Pour cela, on considère l'application $\theta : \mathbb{C}_n[x_1, x_2, x_3] \longrightarrow \mathbb{C}_{n-2}[x_1, x_2, x_3]$ définie par

$$\theta\left(x_1^{k_1} x_2^{k_2} x_3^{k_3}\right) = k_1(k_1-1)x_1^{k_1-2} x_2^{k_2} x_3^{k_3} + k_2 k_3 x_1^{k_1} x_2^{k_2-1} x_3^{k_3-1}.$$

L'application θ est surjective. Il faut maintenant montrer que $\ker\theta = \Gamma_{2n}^{(n)}$. Comme

$$\dim\, \mathbb{C}_k[x_1, x_2, x_3] = \frac{1}{2}\,(k+1)(k+2), \qquad k \in \mathbb{N},$$

on déduit que $\dim(\ker\theta) = 2n+1 = \dim(\Gamma_{2n}^{(n)})$. Montrons que $\Gamma_{2n}^{(n)}$ est inclus dans $\ker\theta$. Par égalité des dimensions, on aura alors $\ker\theta = \Gamma_{2n}^{(n)}$, ce qui terminera la preuve de (B.4.3). Montrons que

$$\theta(\tilde{\Phi}(\tilde{F})^l(x_2^n)) = 0, \tag{B.4.4}$$

pour $0 \le l \le 2n$. Supposons que pour un certain $l \in [\![0, 2n-1]\!]$, (B.4.4) soit vérifié. Alors, on obtient

$$(2m-l)(2m-1-l)\,\alpha_{m,l} = 2(l-m+1)\,\alpha_{m-1,l}, \tag{B.4.5}$$

pour $[(l{+}3)/2] \le m \le \min(l,n)$. On déduit alors de (B.4.2) et de (B.4.5) que $\theta(\tilde{\Phi}(\tilde{F})^{l+1}(x_2^n)) = 0$ et donc que $\ker\theta = \Gamma_{2n}^{(n)}$.

Déterminons maintenant l'expression des coefficients $\alpha_{m,l}$. Pour $l \le n$, on a $\alpha_{l,l} = 1$. On déduit alors de (B.4.5) que

$$\alpha_{m,l} = \frac{l!}{2^{l-m}(l-m)!(2m-l)!}, \qquad \text{pour} \quad [(l+1)/2] \le m \le l.$$

Pour $l > n$, on utilise (B.4.2) pour déduire, par récurrence, que l'expression ci-dessus est encore valable. Ainsi, les

$$\left(\sum_{m=[(l+1)/2]}^{\min(l,n)} \frac{(-1)^{m+l} n!\, l!}{(n-m)!(l-m)!(2m-l)!} x_1^{2m-l} x_2^{n-m}\, x_3^{l-m}\right)_{0\le l\le 2n},$$

forment une base de l'espace vectoriel $\Gamma_{2n}^{(n)}$.

Par récurrence immédiate, (B.4.3) implique que

$$\mathbb{C}_n[x_1, x_2, x_3] = \bigoplus_{j=0}^{[n/2]} \Gamma_{2n-4j}^{(n)}.$$

Cherchons une base de chaque représentation irréductible $\Gamma_{2n-4j}^{(n)}$. On vérifie facilement que, pour chaque $0 \le j \le [n/2]$, $x_2^{n-2j}(x_1^2 + 4x_2x_3)^j$ est un vecteur de plus haut poids associé au poids $2n-4j$. De plus, on a $\tilde{\Phi}(\tilde{F})(x_1^2 + 4x_2x_3) = 0$ et donc, par récurrence immédiate,

$$\tilde{\Phi}(\tilde{F})^k(x_2^{n-2j}(x_1^2 + 4x_2x_3)^j) = (x_1^2 + 4x_2x_3)^j\, \tilde{\Phi}(\tilde{F})^k(x_2^{n-2j}),$$

pour $0 \le k \le 2n-4j$. Ainsi, les

$$(x_1^2 + 4x_2x_3)^j \sum_{m=[(l+1)/2]}^{\min(l,n-2j)} \frac{(-1)^{m+l}(n-2j)!\, l!}{(n-2j-m)!(l-m)!(2m-l)!}\, x_1^{2m-l} x_2^{n-2j-m} x_3^{l-m},$$

pour $0 \leq l \leq 2n-4j$ forment une base de la représentation irréductible $\Gamma^{(n)}_{2n-4j}$.

Il reste maintenant à revenir aux variables $(y_k)_{1 \leq k \leq 3}$. On a

$$x_1 = y_1, \qquad 2x_2 = y_2 - iy_3 \qquad \text{et} \qquad 2x_3 = y_2 + iy_3.$$

Ce qui permet de déduire que (B.4.1) forme une base de la représentation irréductible $\Gamma^{(n)}_{2n-4j}$ de $(\tilde{\Phi}, \mathbb{C}_n[y_1, y_2, y_3])$ pour $0 \leq j \leq [n/2]$. □

On déduit alors le Théorème 4.6.5 du Théorème B.4.2 en suivant la même démarche que dans la Section 4.6.

Bibliographie

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Partie III

Equation de Kac avec thermostat

Dans cette troisième partie, nous étudions les états stationnaires d'une équation de Kac avec thermostat dans le cas où la section efficace est non-intégrable. Ce travail est issu d'une collaboration avec Bernt Wennberg et Yosief Wondmagegne initiée lors de mon séjour en Suède au laboratoire de Mathématiques de Chalmers à Göteborg.

Chapitre 5

Stationary states for the non cut-off Kac equation with a Gaussian thermostat

Travail en cours en collaboration avec Bernt Wennberg et Yosief Wondmagegne.

Abstract

We study the stationary states of a Kac equation with a Gaussian thermostat in the case of a non cut-off cross section. We investigate the existence, smoothness and uniqueness of the stationary states. The theoretical results are illustrated by some numerical simulations.

5.1 Introduction

We consider the non cut-off Kac equation with a thermostated force field

$$\partial_t f + E\,\partial_v((1-\zeta(t)v)f) = Q(f,f), \tag{5.1.1}$$

where $\zeta(t) = \int_{\mathbb{R}} v f(t,v)\,dv$ and

$$Q(f,f)(t,v) = \int_{\mathbb{R}}\int_{-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\,b(\theta)\,d\theta\,dv_*, \tag{5.1.2}$$

with

$$v' = v\cos\theta - v_*\sin\theta, \qquad v'_* = v\sin\theta + v_*\cos\theta,$$

and

$$b(\theta) = |\theta|^{-1-\alpha}, \qquad \theta\in(-\pi,\pi), \qquad \alpha\in(0,2). \tag{5.1.3}$$

The right-hand side is the collision term in Mark Kac's one-dimensional caricature of the Boltzmann equation, and the left-hand side comes from a thermostated force field, which we describe next.

Kac's original equation is derived from the evolution of a stochastic N-particle system, in which the velocity of an individual particle is one-dimensional, and the positions are neglected (see [5]). The system is energy conserving, and therefore the phase space is $\mathbb{S}^{N-1}$. In the original model collisions are modelled by random rotations of randomly chosen pairs of velocities: $(v_j, v_k) \mapsto (v_j\cos\theta - v_k\sin\theta, v_j\sin\theta + v_k\cos\theta)$, and originally θ was chosen uniformly in $[-\pi,\pi]$ (corresponding to $b(\theta) = 1/(2\pi)$ in (5.1.2)), and the intervals between collisions were taken to be exponentially distributed, with a parameter proportional to N.

If the particles are also accelerated by a force field, $dv_j/dt = E$, the system is no longer conservative, but energy conservation can be recovered by projecting the complete force field onto the tangent space of $\mathbb{S}^{N-1}$. This construction is known as a Gaussian isokinetic thermostat, and has been applied in many fields of statistical physics and molecular dynamics, as a model for non equilibrium steady states (see [2, 6, 7, 8] and the references therein).

With the thermostated field and the collisions, a phase space density will evolve according to the following master equation, in which it is assumed that $\sum_{i=1}^N v_i^2 = N$:

$$\partial_t\psi_N(t,\mathbf{V}) + E\sum_{j=1}^N \frac{\partial}{\partial v_j}\left((1-Jv_j)\,\psi_N(t,\mathbf{V})\right)$$
$$= \binom{N}{2}^{-1} \sum_{1\le i<j\le N} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\psi_N\big(t,R_{ij}(\theta)\mathbf{V}\big) - \psi_N(t,\mathbf{V})\Big)\,d\theta, \tag{5.1.4}$$

where $\mathbf{V} = (v_1,...,v_N)$, $NJ = \sum v_j$ and $R_{ij}(\theta)\mathbf{V} = (v_1,\ldots,v'_i,\ldots,v'_j,\ldots,v_N)$.

Equation (5.1.1), with $b(\theta) = 1/(2\pi)$ is obtained by computing the one particle marginals of the solution ψ_N to (5.1.4), and then letting N go to infinity. In that setting, (5.1.1) was considered first in [9], and the principal result is that the equation possesses a stationary solution, and that depending on the field strength E, the stationary solution may be either continuous, or have a power-like singularity. The reason is that there is a competition between the force field, which tries to concentrate the distribution function on a Dirac mass at $v = 1$, and the collision term which drives the distribution function towards a centered Maxwellian. Details of the derivation of (5.1.1) and some generalizations may be found in [10], which also deals with the time dependent problem.

A natural generalization of this is to replace the distribution of rotation angles by a density $b(\theta)$, and Desvillettes [3] introduced a model corresponding to the Boltzmann equation for non-cutoff molecules, in which $b(\theta) \sim |\theta|^{-1-\alpha}$ with $0 < \alpha < 2$. In this case $b(\theta)$ is not integrable, and the collision frequency is infinite. However, for any $0 < \theta_1 < \theta_2$, $\int_{\theta_1 \leq |\theta| \leq \theta_2} b(\theta)\, d\theta$ is finite, and corresponds to the expected frequency of jumps with θ in the given interval. In this way, the collisions still form a Poisson process. Desvillettes, who was the first to consider this (with no force field), proved that the collision operator is smoothing in this case, and that the solutions to the time dependent problem immediately become smooth, much like in the heat equation. Also for the Boltzmann equation, the non-cutoff collision operator has a smoothing effect (see [1, 4]).

In this paper we consider the stationary case of (5.1.1):

$$E\, \frac{d}{dv}((1 - \zeta v) f(v)) = Q(f, f)(v), \qquad v \in \mathbb{R}. \tag{5.1.5}$$

We are interested in the question as to whether the regularizing effect of the non-cutoff collision operator is enough to prevent also a very strong force field to yield a singularity of the stationary solution. We show that this is the case, and in fact, it is not a surprising result. While with the cutoff collision operator, there are two distinct time scales that can be directly compared with each other: the mean time between collisions, and a time scale related to the acceleration of particles, only the time scale related to the acceleration remains in the non-cutoff case.

We now give a relevant definition of solutions in the non-cutoff situation, and state the main result of the paper.

Definition 5.1.1 *Assume that b satisfies (5.1.3). A function $f \in L^1_2(\mathbb{R})$ is said to be a weak solution to (5.1.5)-(5.1.2) if it satisfies*

$$\int_{\mathbb{R}} (\zeta\, v - 1)\, f(v)\, \psi'(v)\, dv = \frac{1}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta\, f(v)\, f(v_*)\, dv\, dv_*, \tag{5.1.6}$$

for every $\psi \in \mathcal{C}^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Our main result is the following.

Theorem 5.1.2 *Assume that b satisfies (5.1.3). For all field strengths $E > 0$, there exists a unique weak solution f to (5.1.5)-(5.1.2) such that moments of any order of f are finite and*

$$\int_{\mathbb{R}} f(v)\, dv = 1. \tag{5.1.7}$$

Moreover, $f \in \mathcal{C}^\infty(\mathbb{R})$.

The paper is organized as follows. First, in Section 5.2, we show that a solution to equation (5.1.5)-(5.1.2) exists if the cross-section b is supposed to be Lipschitz continuous on $[-\pi, \pi]$. The proof is to a large extent an adaptation to the present case of the techniques used in [9]. As with the Boltzmann equation for Maxwellian molecules, much is simplified by the possibility to compute moments exactly.

We then deduce in Section 5.3.1 the existence of a solution to (5.1.5)-(5.1.2) for a cross-section b satisfying (5.1.3). The smoothness of such a solution is investigated in Section 5.3.2 by the use of Fourier transform techniques, much like in [3]. Section 5.3.3 is then devoted to the proof of the uniqueness part of Theorem 5.1.2.

We also illustrate, in Section 5.4, the theoretical results by some numerical simulations, obtained by solving the equation satisfied by the Fourier transform.

5.2 Cut-off case

We consider here the stationary equation (5.1.5)-(5.1.2) when $b : [-\pi, \pi] \to \mathbb{R}$ is an even and Lipschitz continuous function. Without loss of generality, one can assume that $\int_{\mathbb{R}} f(v)\, dv = 1$. Then, the collision operator reads

$$Q(f, f)(v) = Q^+(f, f)(v) - \|b\|_{L^1} f(v), \qquad v \in \mathbb{R},$$

where

$$Q^+(f, f)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(v')\, f(v'_*)\, b(\theta)\, d\theta\, dv_*. \tag{5.2.1}$$

Since

$$\int_{\mathbb{R}} Q^+(f, f)(v)\, v\, dv = \zeta \int_{-\pi}^{\pi} \cos\theta\, b(\theta)\, d\theta,$$

we obtain, by multiplying (5.1.5) by v and integrating, that ζ satisfies $\zeta^2 + (K/E)\,\zeta - 1 = 0$, where $K := \int_{-\pi}^{\pi} (1 - \cos\theta)\, b(\theta)\, d\theta$. We then deduce that

$$\zeta = \frac{\sqrt{K^2 + 4E^2} - K}{2E}$$

is the only root that allows $\int_{\mathbb{R}} v^2 f(v)\, dv \leq 1$. We set $\kappa = 1/\zeta$ and

$$\gamma = \frac{\|b\|_{L^1}}{E\,\zeta} - 1.$$

Dividing (5.1.5) by $(v - \kappa)|v - \kappa|^{\gamma}$, we obtain, for $v \neq \kappa$,

$$\frac{d}{dv}\left(\frac{1}{|v - \kappa|^{\gamma}}\, f(v)\right) = -\,\frac{\gamma + 1}{\|b\|_{L^1}}\, \frac{1}{(v - \kappa)|v - \kappa|^{\gamma}}\, Q^+(f, f)(v).$$

Then, any solution to (5.1.5)-(5.1.2) with $\int_{\mathbb{R}} f(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2\, f(v)\, dv = 1$ satisfies

$$f(v) = \mathcal{A}(f)(v), \qquad v \in \mathbb{R} \backslash \{\kappa\}, \tag{5.2.2}$$

where

$$\mathcal{A}(f)(v) = \frac{\gamma+1}{\|b\|_{L^1}}\, |v-\kappa|^\gamma \begin{cases} \displaystyle\int_{-\infty}^{v} \frac{1}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, dw & (v<\kappa) \\ \displaystyle\int_{v}^{\infty} \frac{1}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, dw & (v>\kappa) \end{cases}$$

Theorem 5.2.1 *Let $b : [-\pi,\pi] \to \mathbb{R}$ be an even and Lipschitz continuous function. For all field strengths $E > 0$, there exists a solution f to (5.2.2) such that $f \in \mathcal{C}(\mathbb{R}\backslash\{\kappa\})$, moments of any order of f are finite,*

$$\int_{\mathbb{R}} f(v)\, dv = 1, \qquad \int_{\mathbb{R}} f(v)\, v\, dv = \zeta \qquad \text{and} \qquad \int_{\mathbb{R}} v^2\, f(v)\, dv = 1.$$

Moreover, for $\gamma > 0$, $f \in \mathcal{C}(\mathbb{R})$ and $Q^+(f,f) \in \mathcal{C}(\mathbb{R})$.

The proof of Theorem 5.2.1 is similar to that of [9, Theorem 1]. A solution to (5.2.2) is obtained by passing to the limit in the sequence generated by the iteration

$$f_{n+1} = \mathcal{A}(f_n). \tag{5.2.3}$$

Therefore, some estimates on $\mathcal{A}$ are needed. We first define

$$\Lambda(\psi)(w) = \frac{\gamma+1}{|w-\kappa|^{\gamma+1}} \int_{\kappa}^{w} (v-\kappa)\, |v-\kappa|^{\gamma-1}\, \psi(v)\, dv, \qquad w \in \mathbb{R},$$

and

$$\overline{\Lambda}(\psi)(v,v_*) = \int_{-\pi}^{\pi} \Lambda(\psi)(v')\, \frac{b(\theta)}{\|b\|_{L^1}}\, d\theta, \qquad (v,v_*) \in \mathbb{R}^2.$$

For $\psi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} \int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv \;=\; & \frac{\gamma+1}{\|b\|_{L^1}} \int_{-\infty}^{\kappa} \int_{-\infty}^{v} \frac{|v-\kappa|^\gamma}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, \psi(v)\, dw\, dv \\ & + \frac{\gamma+1}{\|b\|_{L^1}} \int_{\kappa}^{\infty} \int_{v}^{\infty} \frac{|v-\kappa|^\gamma}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, \psi(v)\, dw\, dv. \end{aligned}$$

Thus, changing the order of integration, we obtain

$$\int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv = \int_{\mathbb{R}} \Lambda(\psi)(w)\, Q^+(f,f)(w)\, \frac{dw}{\|b\|_{L^1}}.$$

By (5.2.1), we deduce that

$$\int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv = \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, f(v)\, f(v_*)\, dv\, dv_*.$$

Since, for any bounded measure μ, $v \mapsto \int_{\mathbb{R}} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)$ is a bounded and continuous function, the mapping $f \mapsto \mathcal{A}(f)$ extends to a mapping $\mu \mapsto \mathcal{A}(\mu)$ on the space of bounded measures. A first step in the proof of Theorem 5.2.1 consists in a computation of the moments of $\mathcal{A}(\mu)$.

Lemma 5.2.2 *Let $m \in \mathbb{N}_*$. Then, for any measure μ satisfying $\int_{\mathbb{R}} \mu(dv) = 1$, we have*

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{A}(\mu)(dv) &= 1, \\
\int_{\mathbb{R}} v\, \mathcal{A}(\mu)(dv) &= \int_{\mathbb{R}} v\, \mu(dv) + \frac{1}{\gamma+2}\left(1 - \frac{K\,(\gamma+3)}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right), \\
\int_{\mathbb{R}} v^2 \mathcal{A}(\mu)(dv) &= \int_{\mathbb{R}} v^2 \mu(dv) + \frac{2\,\kappa}{\gamma+2}\left(1 - \frac{K}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right) \\
&\quad + \frac{2}{\gamma+3}\left[\left(1 - \int_{\mathbb{R}} v^2 \mu(dv)\right) - 2\,\kappa\left(1 - \frac{K}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right)\right], \\
\int_{\mathbb{R}} v^m \mathcal{A}(\mu)(dv) &= \left(1 - \frac{m}{m+\gamma+1}\right)\int_{-\pi}^{\pi} (\cos^m\theta + \sin^m\theta)\, \frac{b(\theta)}{\|b\|_{L^1}}\, d\theta \int_{\mathbb{R}} v^m \mu(dv) \\
&\quad + M_{m-1},
\end{aligned} \tag{5.2.4}$$

where M_{m-1} only depends on moments of order $\leq m-1$.

Proof. The proof of this lemma is similar to that of [9, Lemma 2]. Let $m \in \mathbb{N}_*$. For $\psi(v) = v^m$, we have

$$\begin{aligned}
\Lambda(\psi)(v) &= v^m - \frac{m}{|v-\kappa|^{\gamma+1}}\int_{\kappa}^{v} |w-\kappa|^{\gamma+1} w^{m-1}\, dw \\
&= v^m - \frac{m}{\gamma+m+1}(v-\kappa)^m + \frac{m}{|v-\kappa|^{\gamma+1}}\int_{\kappa}^{v} |w-\kappa|^{\gamma+1} P_{m-2}(w)\, dw
\end{aligned}$$

where $P_{m-2}(w) = (w-\kappa)^{m-1} - w^{m-1}$ is a polynomial of degree $m-2$. For $m = 1$ and $m = 2$, we obtain, respectively,

$$\begin{aligned}
\Lambda(\psi)(v) &= \left(1 - \frac{1}{\gamma+2}\right) v + \frac{\kappa}{\gamma+2}, \\
\Lambda(\psi)(v) &= \left(1 - \frac{2}{\gamma+3}\right) v^2 + \left(\frac{-2\kappa}{\gamma+2} + \frac{4\kappa}{\gamma+3}\right) v + \left(\frac{2\kappa^2}{\gamma+2} - \frac{2\kappa^2}{\gamma+3}\right).
\end{aligned}$$

Then, Lemma 5.2.2 follows easily. □

Remarque 5.2.3 *Lemma 5.2.2 ensures that the iteration (5.2.3) gives a tight set of unit measures. By weak compactness, there exists a measure μ such that, up to an extraction, $f_n \rightharpoonup \mu$.*

We now investigate the singularity of $\mathcal{A}(\mu)$ at $v = \kappa$. In the rest of this paper, C denotes a positive constant that may vary from one step to the other.

Lemma 5.2.4 *Let us assume that $f \in L^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} v^2 f(v)\, dv < \infty$ and $f(v) \leq C'|v-\kappa|^{-1}$, for some constant $C' > 0$. Then there is a constant $C > 0$ such that*

$$Q^+(f,f)(v) \leq C(1 + (\log|v-\kappa|)^2),$$

in a neighbourhood of $v = \kappa$. The constant C depends only on κ and $\|b\|_{L^\infty}$.

Proof. Since $b \in L^\infty(-\pi,\pi)$, we have

$$Q^+(f,f)(v) \leq \|b\|_{L^\infty} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(v')\, f(v'_*)\, d\theta\, dv_*.$$

The remainder of the proof is then similar to that of [9, Lemma 3]. □

Lemma 5.2.5 *Let $f \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} v^2 f(v)\, dv < \infty$ and $f(v) \leq C(1+(\log|v-\kappa|)^2)$ for some constant $C > 0$. Then, $Q^+(f,f)$ is bounded in a neighbourhood of $v = \kappa$.*

Moreover, if $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then $Q^+(f,f)$ is Hölder-continuous with exponent $1/2$ on each ball $B(0,R)$ with centre 0 and radius R. The constant of Hölder-continuity depends only on R, $\|f\|_{L^\infty}$, $\|b\|_{L^\infty}$ and the Lipschitz constant of b.

Proof. The L^∞-bound of the cross-section b enables us to proceed as in [9, Lemma 4] to prove the boundedness of $Q^+(f,f)$ in a neighbourhood of $v = \kappa$. However, the proof of the Hölder-continuity of $Q^+(f,f)$ has to be slightly modified.

For $w \in \mathbb{R}$, we set $\Omega := \{(u,u_*) \in \mathbb{R}^2 : u^2 + u_*^2 > w^2\}$. By a change of variables, the collision operator Q^+ can be rewritten as

$$Q^+(f,f)(w) = \int_{\mathbb{R}^2} \frac{\mathbf{1}_\Omega(u,u_*)}{\sqrt{u^2+u_*^2-w^2}}\, f(u)\, f(u_*)\, \bar{b}(u,u_*,w)\, du\, du_*,$$

where

$$\bar{b}(u,u_*,w) = b\left(\mathrm{Arccos}\left(\frac{uw + u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) + b\left(\mathrm{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right).$$

Thus, for $|v| < |w| < M/2$,

$$|Q^+(f,f)(w) - Q^+(f,f)(v)| \leq \int_{\mathbb{R}^2} f(u)\, f(u_*)\, \mathbf{1}_{u^2+u_*^2>w^2} \frac{|\bar{b}(u,u_*,w) - \bar{b}(u,u_*,v)|}{\sqrt{u^2+u_*^2-w^2}}\, du\, du_* + \|f\|_{L^\infty}^2 \|b\|_{L^\infty} \int_{\mathbb{R}^2} \left| \frac{\mathbf{1}_{u^2+u_*^2>w^2}}{\sqrt{u^2+u_*^2-w^2}} - \frac{\mathbf{1}_{u^2+u_*^2>v^2}}{\sqrt{u^2+u_*^2-v^2}} \right| du\, du_*.$$

The second integral in the right hand side can be handled as in [9, Lemma 4]. Therefore, we only consider the first one. Since b is Lipschitz continuous, we have

$$\begin{aligned}
&\left| b\left(\operatorname{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) - b\left(\operatorname{Arccos}\left(\frac{uv - u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right)\right| \\
&\qquad \le C \left|\operatorname{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right) - \operatorname{Arccos}\left(\frac{uv - u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right| \\
&\qquad \le C \left| \int_v^w \frac{u\sqrt{u^2+u_*^2-x^2}+u_* x}{|u\sqrt{u^2+u_*^2-x^2}+u_* x|} \; \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| \\
&\qquad \le C \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left| b\left(\operatorname{Arccos}\left(\frac{uw + u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) - b\left(\operatorname{Arccos}\left(\frac{uv + u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right)\right| \\
&\qquad\qquad\qquad\qquad \le C \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right|.
\end{aligned}$$

We set $G = \{(u,u_*) \in \Omega : |u| \le M,\ |u_*| \le M\}$ and $B = \Omega \setminus G$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*)\, \mathbf{1}_{u^2+u_*^2>w^2} \; \frac{|\bar b(u,u_*,w) - \bar b(u,u_*,v)|}{\sqrt{u^2+u_*^2-w^2}} \; du\, du_* \\
&\qquad \le 2\,C \int_{\mathbb{R}^2} f(u)\, f(u_*) \; \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}} \; \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| \; du\, du_* \\
&\qquad + 2\,C \int_{\mathbb{R}^2} f(u)\, f(u_*) \; \frac{\mathbf{1}_B}{\sqrt{u^2+u_*^2-w^2}} \; \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| \; du\, du_*.
\end{aligned}$$

But, for $(u,u_*) \in B$, we have $u^2+u_*^2-w^2 \ge 3\,M^2/4$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*) \; \frac{\mathbf{1}_B}{\sqrt{u^2+u_*^2-w^2}} \; \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| \; du\, du_* \\
&\qquad \le \int_{\mathbb{R}^2} f(u)\, f(u_*) \; \frac{\mathbf{1}_B}{u^2+u_*^2-w^2} \; |w-v| \, du\, du_* \le \frac{4}{3M^2} \; \|f\|_{L^1}^2 \, |w-v|.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| \;&\le\; \frac{1}{(u^2+u_*^2-w^2)^{1/4}} \; |w-v|^{1/2} \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right|^{1/2} \\
&\le\; \frac{\sqrt{\pi}}{(u^2+u_*^2-w^2)^{1/4}} \; |w-v|^{1/2},
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*)\, \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}} \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| du\, du_* \\
&\quad \le \sqrt{\pi}\, |w-v|^{1/2} \int_{\mathbb{R}^2} f(u)\, f(u_*)\, \frac{\mathbf{1}_G}{(u^2+u_*^2-w^2)^{3/4}}\, du\, du_* \\
&\quad \le \sqrt{\pi}\, |w-v|^{1/2} \left(\int_{\mathbb{R}^2} f(u)^p\, f(u_*)^p\, du\, du_* \right)^{1/p} \left(\int_{\mathbb{R}^2} \frac{\mathbf{1}_G}{(u^2+u_*^2-w^2)^{3q/4}}\, du\, du_* \right)^{1/q}.
\end{aligned}$$

For $q = 5/4$ and $p = 5$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u) f(u_*)\, \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}} \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| du\, du_* \\
&\qquad\qquad \le \sqrt{\pi}\, (16\pi)^{4/5}\, (2M^2)^{1/20}\, \|f\|_{L^\infty}^{8/5}\, |w-v|^{1/2}.
\end{aligned}$$

□

Lemma 5.2.6 *Let $g \in L^1([0,\infty))$ be a non-negative function satisfying $g(v) \le C(1 + (\log v)^2)$ for some constant $C > 0$. For $v > 0$, we set*

$$F(v) = v^\gamma \int_v^\infty \frac{1}{w^{\gamma+1}}\, g(w)\, dw.$$

Then, for $\gamma > 0$, $F(v) \le C(1+(\log v)^2)$ and, if we also assume that g is Hölder-continuous in $v \in [0,\infty)$ then F belongs to $\mathcal{C}([0,\infty))$.

Proof of Theorem 5.2.1. Let $f_0 \in L^1(\mathbb{R})$ be a non-negative function that has finite moments of any order and that satisfies

$$\int f_0(v)\, dv = 1, \qquad \int v\, f_0(v)\, dv = \zeta \qquad \text{and} \qquad \int v^2\, f_0(v)\, dv = 1.$$

We consider the associated sequence $(f_n)_{n\in\mathbb{N}}$ generated by the iteration (5.2.3). As stated in Remark 5.2.3, up to an extraction, this sequence converges to a measure μ. We deduce from Lemma 5.2.2 that μ satisfies

$$\int_{\mathbb{R}} \mu(dv) = 1, \qquad \int_{\mathbb{R}} v\, \mu(dv) = \zeta \qquad \text{and} \qquad \int_{\mathbb{R}} v^2\, \mu(dv) = 1.$$

This measure has a non-negative density $f \in L^1(\mathbb{R})$. By induction, we infer from (5.2.4) that each moment of f_n is bounded, independently of n and then that each moment of f is bounded. We now pass to the limit in (5.2.3). Since $v \mapsto \int_{\mathbb{R}} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)$ is a bounded and continuous function, we have

$$\int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, f_n(v)\, dv \longrightarrow \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, \mu(dv),$$

as $n \to \infty$. Then, since $\int_{\mathbb{R}} f_n(v)\, v^2\, dv = \int_{\mathbb{R}} v^2\, \mu(dv) = 1$ and $(v, v_*) \mapsto \overline{\Lambda}(\psi)(v, v_*)$ is a continuous function, we deduce that

$$\left| \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v, v_*)\, f_n(v_*)\, f_n(v)\, dv_*\, dv - \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v, v_*)\, \mu(dv_*)\, f_n(v)\, dv \right| \longrightarrow 0,$$

as $n \to \infty$. It then follows readily that

$$\int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v, v_*)\, f_n(v_*)\, f_n(v)\, dv_*\, dv \longrightarrow \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v, v_*)\, \mu(dv_*)\, \mu(dv),$$

as $n \to \infty$. Since $Q_+(f, f) \in L^1(\mathbb{R})$, $\mathcal{A}(f)$ is continuous on $\mathbb{R}\backslash\{\kappa\}$. The proof of the first statement of Theorem 5.2.1 is now complete. Let us consider the case $\gamma > 0$. Let $k \in \mathbb{N}$. By definition of $\mathcal{A}$ and Q^+, we have

$$f_k(v) \leq C|v - \kappa|^{-1}, \qquad v \in \mathbb{R},$$

where C denotes some constant that only depends on E and b. Lemma 5.2.4 implies that

$$Q^+(f_k, f_k)(v) \leq C(1 + (\log|v - \kappa|)^2),$$

in a neighbourhood of $v = \kappa$, where C only depends on E and b. We now deduce from Lemma 5.2.6 that $f_{k+1}(v) \leq C(1 + (\log|v - \kappa|)^2)$. By Lemma 5.2.5, $Q^+(f_{k+1}, f_{k+1})$ is thus bounded in a neighbourhood of $v = \kappa$. It follows from Lemma 5.2.6 that f_{k+2} is bounded. We infer from Lemma 5.2.5 that $Q^+(f_{k+2}, f_{k+2})$ is locally Hölder-continuous. Finally, Lemma 5.2.6 implies that f_{k+3} is a continuous function. Since these estimates do not depend on k, they also hold for the limit f. □

5.3 Non cut-off case

We now consider (5.1.5)-(5.1.2) when the cross section b satisfies (5.1.3) and prove Theorem 5.1.2.

5.3.1 Existence

We first investigate the existence of a weak solution to (5.1.5)-(5.1.2) and prove the following theorem.

Theorem 5.3.1 *Assume that b satisfies (5.1.3). For all field strengths $E > 0$, there exists a weak solution f to (5.1.5)-(5.1.2), in the sense of Definition 5.1.1, that satisfies (5.1.7) and such that moments of any order of f are finite.*

Proof. For $n \in \mathbb{N}$, we set $b_n = b \wedge n$, where, for every $c, d \in \mathbb{R}$, $c \wedge d$ denotes the minimum value of c and d. We deal with the associated stationary equation

$$E\, \frac{d}{dv}((1 - \zeta_n v) f_n(v)) = Q_n(f_n, f_n)(v), \qquad v \in \mathbb{R}, \tag{5.3.1}$$

where $\zeta_n = \int_{\mathbb{R}} v f_n(v)\, dv$ and

$$Q_n(f_n, f_n)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f_n(v') f_n(v'_*) - f_n(v) f_n(v_*))\, b_n(\theta)\, d\theta\, dv_*.$$

As previously, we obtain that

$$\zeta_n = \frac{\sqrt{K_n^2 + 4E^2} - K_n}{2E},$$

with $K_n := \int_{-\pi}^{\pi} (1 - \cos\theta)\, b_n(\theta)\, d\theta$. We set $\kappa_n = 1/\zeta_n$ and

$$\gamma_n = \frac{\|b_n\|_{L^1}}{E\,\zeta_n} - 1.$$

Then, any solution to (5.3.1) with $\int_{\mathbb{R}} f_n(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2 f_n(v)\, dv = 1$ satisfies

$$f_n = \mathcal{A}_n(f_n), \tag{5.3.2}$$

where

$$\mathcal{A}_n(f_n)(v) = \frac{\gamma_n + 1}{\|b_n\|_{L^1}}\, |v - \kappa_n|^{\gamma_n} \begin{cases} \displaystyle\int_{-\infty}^{v} \frac{1}{|w - \kappa_n|^{\gamma_n + 1}}\, Q_n^+(f_n, f_n)(w)\, dw & (v < \kappa_n) \\ \displaystyle\int_{v}^{\infty} \frac{1}{|w - \kappa_n|^{\gamma_n + 1}}\, Q_n^+(f_n, f_n)(w)\, dw & (v > \kappa_n) \end{cases}$$

with

$$Q_n^+(f_n, f_n)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} f_n(v')\, f_n(v'_*)\, b_n(\theta)\, d\theta\, dv_*.$$

The existence of a solution f_n to (5.3.2) satisfying $\int_{\mathbb{R}} f_n(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2 f_n(v)\, dv = 1$ follows from Theorem 5.2.1. Moreover, since γ_n goes to infinity, there exists n_0 such that, for each $n \geq n_0$, γ_n is positive. Then, for each $n \geq n_0$, the functions f_n and $Q_n^+(f_n, f_n)$ are continuous. Thus, for each $n \geq n_0$, $f_n \in \mathcal{C}^1(\mathbb{R}\backslash\{\kappa_n\})$ and, for $v \neq \kappa_n$, we have

$$\frac{d}{dv}((v - \kappa_n) f_n(v)) = -\frac{\kappa_n}{E}\, Q_n(f_n, f_n)(v).$$

For $\psi \in \mathcal{D}(\mathbb{R})$, we deduce that

$$\int_{\mathbb{R}} (v - \kappa_n)\, f_n(v)\, \psi'(v)\, dv = \frac{\kappa_n}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b_n(\theta)\, d\theta\, f_n(v)\, f_n(v_*)\, dv\, dv_*. \tag{5.3.3}$$

Since f_n has finite moments of any order, classical truncation argument ensures that, for every integer $m \geq 3$,

$$\left(m + \frac{\kappa_n}{E} \int_{-\pi}^{\pi} (1 - \cos^m\theta - \sin^m\theta)\, b_n(\theta)\, d\theta\right) \int_{\mathbb{R}} v^m f_n(v)\, dv = m\, \kappa_n \int_{\mathbb{R}} v^{m-1} f_n(v)\, dv$$
$$+ \frac{\kappa_n}{E} \sum_{k=1}^{[(m-1)/2]} \binom{m}{2k} \int_{-\pi}^{\pi} (\cos\theta)^{m-2k} (\sin\theta)^{2k}\, b_n(\theta)\, d\theta \int_{\mathbb{R}} v^{2k} f_n(v)\, dv \int_{\mathbb{R}} v^{m-2k} f_n(v)\, dv,$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$. It then follows by induction that, for each $m \in \mathbb{N}$, there exists a constant C_m independent of n such that

$$\int_{\mathbb{R}} v^m f_n(v)\, dv \le C_m. \tag{5.3.4}$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ is a tight set of unit measures and, by weak compactness, there exists a measure μ such that, up to an extraction, $f_n \rightharpoonup \mu$. Then, we have

$$\int_{\mathbb{R}} \mu(dv) = 1, \qquad \int_{\mathbb{R}} v^2 \mu(dv) = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} v^m \mu(dv) \le C_m.$$

Moreover, the measure μ has a non-negative density $f \in L^1(\mathbb{R})$.

Let us now pass to the limit in (5.3.3). We set, for $(v, v_*) \in \mathbb{R}^2$,

$$\begin{aligned}
\Xi_n(v, v_*) &= \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b_n(\theta)\, d\theta \\
&= \frac{1}{2}\, v_*^2 \int_{-\pi}^{\pi} \int_{-1}^{1} (1-|r|)\, \psi''(v\cos\theta + r v_* \sin\theta)\, dr \sin^2\theta\, b_n(\theta)\, d\theta \\
&\quad - v \int_{-\pi}^{\pi} \int_{0}^{1} \psi'(v + rv\,(\cos\theta - 1))\, dr\, (1-\cos\theta)\, b_n(\theta)\, d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\Xi(v, v_*) &= \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta \\
&= \frac{1}{2}\, v_*^2 \int_{-\pi}^{\pi} \int_{-1}^{1} (1-|r|)\, \psi''(v\cos\theta + r v_* \sin\theta)\, dr \sin^2\theta\, b(\theta)\, d\theta \\
&\quad - v \int_{-\pi}^{\pi} \int_{0}^{1} \psi'(v + rv\,(\cos\theta - 1))\, dr\, (1-\cos\theta)\, b(\theta)\, d\theta.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
|\, \Xi_n(v, v_*) - \Xi(v, v_*)\,| &\le 2\, v_*^2\, \|\psi''\|_{L^\infty} \int_0^{n^{-1/(1+\alpha)}} \sin^2\theta\, b(\theta)\, d\theta \\
&\quad + 2\,|v|\, \|\psi'\|_{L^\infty} \int_0^{n^{-1/(1+\alpha)}} (1-\cos\theta)\, b(\theta)\, d\theta,
\end{aligned} \tag{5.3.5}$$

and

$$|\, \Xi_n(v, v_*)\,| \le C\, \|\psi\|_{W^{2,\infty}}\, (v_*^2 + |v|), \qquad |\, \Xi(v, v_*)\,| \le C\, \|\psi\|_{W^{2,\infty}}\, (v_*^2 + |v|). \tag{5.3.6}$$

We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \Xi_n(v, v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \mu(dv)\, \mu(dv_*) \right| \\
&\qquad \le \left| \int_{\mathbb{R}^2} (\Xi_n(v, v_*) - \Xi(v, v_*))\, f_n(v)\, f_n(v_*)\, dv\, dv_* \right| && (5.3.7) \\
&\qquad + \left| \int_{\mathbb{R}^2} \Xi(v, v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \mu(dv)\, \mu(dv_*) \right|. && (5.3.8)
\end{aligned}$$

The estimates (5.3.4) and (5.3.5) ensure that the integral (5.3.7) tends to 0 as $n \to +\infty$. It thus remains to consider (5.3.8). Let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that $\eta(v) = 1$ if $|v| \leq 1$ and $\eta(v) = 0$ if $|v| \geq 2$. Denote by η_R the function defined by $\eta_R(v) = \eta(v/R)$, for every $v \in \mathbb{R}$. Then,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \mu(dv)\, \mu(dv_*)\right| \\
&\leq \left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv)\, \mu(dv_*)\right| \\
&+ \int_{\mathbb{R}^2} (1-\eta_R(v_*))\, |\Xi(v,v_*)|\, f_n(v)\, f_n(v_*)\, dv\, dv_* + \int_{\mathbb{R}^2} (1-\eta_R(v_*))\, |\Xi(v,v_*)|\, \mu(dv)\, \mu(dv_*) \\
&+ \int_{\mathbb{R}^2} (1-\eta_R(v))\, |\Xi(v,v_*)|\, f_n(v)\, f_n(v_*)\, dv\, dv_* + \int_{\mathbb{R}^2} (1-\eta_R(v))\, |\Xi(v,v_*)|\, \mu(dv)\, \mu(dv_*).
\end{aligned}$$

We now infer from (5.3.4) and (5.3.6) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \mu(dv)\, \mu(dv_*)\right| \\
&\leq \left|\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v_*)\, dv_* - \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv_*)\right) f_n(v)\, dv\right| \quad (5.3.9) \\
&+ \left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)\, f_n(v)\, dv - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)\, \mu(dv)\right| \quad (5.3.10) \\
&+ 2\, C\, \|\psi\|_{W^{2,\infty}}\, \frac{3+C_4}{R}.
\end{aligned}$$

Since $(v,v_*) \mapsto \eta_R(v)\, \eta_R(v_*)\, \Xi(v,v_*)$ is a compactly supported continuous function,

$$\int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v_*)\, dv_* \longrightarrow \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv_*),$$

uniformly in v as $n \to +\infty$. Consequently, (5.3.9) tends to 0 as $n \to +\infty$. Moreover, the function $v \mapsto \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)$ belongs to $\mathcal{C}_b(\mathbb{R})$ and thus (5.3.10) also tends to 0 as $n \to +\infty$. Finally, we obtain, for every $\psi \in \mathcal{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} (v-\kappa)\, \psi'(v)\, \mu(dv) = \frac{\kappa}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta\, \mu(dv)\, \mu(dv_*),$$

where

$$\kappa = \frac{1}{\zeta} = \frac{2E}{\sqrt{K^2+4E^2}-K} \qquad \text{with} \qquad K := \int_{-\pi}^{\pi} (1-\cos\theta)\, |\theta|^{-1-\alpha}\, d\theta. \qquad (5.3.11)$$

Therefore, the density f of μ satisfies (5.1.6) for every $\psi \in \mathcal{D}(\mathbb{R})$ and, by density, for every $\psi \in \mathcal{C}^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. □

5.3.2 Smoothness

We now show the following theorem.

Theorem 5.3.2 *Assume that b satisfies (5.1.3). For all field strengths $E > 0$, if f is a weak solution to (5.1.5)-(5.1.2), in the sense of Definition 5.1.1, then $f \in \mathcal{C}^\infty(\mathbb{R})$.*

We consider here the Fourier transform $\hat{f}$ of f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-iv\xi} f(v)\, dv, \qquad \xi \in \mathbb{R}.$$

By (5.1.6), $\hat{f}$ satisfies

$$\hat{f}'(\xi) + i\kappa \hat{f}(\xi) = \frac{\kappa}{E\,\xi} \int_{-\pi}^{\pi} \Big(\hat{f}(\xi\cos\theta)\,\hat{f}(\xi\sin\theta) - \hat{f}(0)\,\hat{f}(\xi)\Big)\, b(\theta)\, d\theta, \tag{5.3.12}$$

where $\kappa = 1/\zeta$. We set, for every $\xi \in \mathbb{R}$,

$$\begin{aligned} q_1(\xi) &= -\frac{1}{2\,E} \int_{-\pi}^{\pi} \Big(\hat{f}(\xi\sin\theta) + \hat{f}(-\xi\sin\theta) - 2\hat{f}(0)\Big)\, b(\theta)\, d\theta, \\ q_2(\xi) &= \frac{\kappa}{E} \int_{-\pi}^{\pi} \hat{f}(\xi\sin\theta)\Big(\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\Big)\, b(\theta)\, d\theta. \end{aligned}$$

Then, (5.3.12) reads

$$\hat{f}'(\xi) + \kappa\left(i + \frac{q_1(\xi)}{\xi}\right)\hat{f}(\xi) = \frac{q_2(\xi)}{\xi}, \qquad \xi \in \mathbb{R}. \tag{5.3.13}$$

Lemma 5.3.3 *There exists $M \geq 1$ such that, for $|\xi| \geq M$,*

$$q_1(\xi) \geq D_f\,|\xi|^\alpha \qquad \textit{with} \quad D_f = \frac{1}{4\,E}\int_{\mathbb{R}} f(v)\,|v|^\alpha dv \int_0^\infty \sin^2(y/2)|y|^{-1-\alpha} dy > 0, \tag{5.3.14}$$

$$|q_2(\xi)| \leq G_f\,|\xi|^{\alpha/2} \quad \textit{with} \quad G_f = \frac{3\,\kappa}{E}\int_{\mathbb{R}} f(v)\,|v|^{\alpha/2} dv \int_0^\infty \left|e^{2iy^2} - 1\right| |y|^{-1-\alpha} dy + 20\,\frac{\kappa}{E}. \tag{5.3.15}$$

Proof. The function q_1 may be written under the form

$$q_1(\xi) = \frac{2}{E}\int_{\mathbb{R}} f(v) \int_{-\pi}^{\pi} \sin^2\left(\frac{v\,\xi\,\sin\theta}{2}\right) |\theta|^{-1-\alpha}\, d\theta\, dv.$$

Thus, $q_1(\xi)$ is real and

$$q_1(\xi) \geq \frac{4}{E}\int_{\mathbb{R}} f(v) \int_0^{\pi/2} \sin^2\left(\frac{v\,\xi\,\sin\theta}{2}\right) |\theta|^{-1-\alpha}\, d\theta\, dv.$$

The successive changes of variables $x = \sin\theta$ and $y = |v\,\xi|\,x$ lead to

$$\begin{aligned} q_1(\xi) &\geq \frac{4}{E}\int_{\mathbb{R}} f(v)\int_0^1 \sin^2\left(\frac{v\,\xi\,x}{2}\right)|2x|^{-1-\alpha}\,dx\,dv \\ &\geq \frac{2^{1-\alpha}}{E}\,|\xi|^\alpha\int_{\mathbb{R}} f(v)\,|v|^\alpha\int_0^{|v\xi|}\sin^2(y/2)\,|y|^{-1-\alpha}\,dy\,dv. \end{aligned}$$

Since we have

$$\int_{\mathbb{R}} f(v)\,|v|^\alpha\int_0^{|v\xi|}\sin^2(y/2)\,|y|^{-1-\alpha}\,dy\,dv \longrightarrow \int_{\mathbb{R}} f(v)\,|v|^\alpha\,dv\int_0^\infty \sin^2(y/2)\,|y|^{-1-\alpha}\,dy,$$

as $|\xi| \to \infty$, we deduce that there exists $M \geq 0$ such that (5.3.14) holds for $|\xi| \geq M$. Since

$$\begin{aligned} \int_{\mathbb{R}} f(v)\,|v|^\alpha\,dv &\geq R^{\alpha-2}\int_{|v|\leq R} f(v)\,|v|^2\,dv \\ &\geq R^{\alpha-2}\left(\int_{\mathbb{R}} f(v)\,|v|^2\,dv - \int_{|v|\geq R} f(v)\,|v|^2\,dv\right), \end{aligned}$$

the constant D_f is positive. As for q_2, we have

$$\begin{aligned} |q_2(\xi)| &\leq \frac{\kappa}{E}\int_{\mathbb{R}}\int_{-\pi}^{\pi} f(v)\left|\,e^{-iv\xi(\cos\theta-1)} - 1\right|\,|\theta|^{-1-\alpha}\,d\theta\,dv \\ &\leq \frac{2\,\kappa}{E}\int_{\mathbb{R}}\int_0^{\pi/2} f(v)\left|\,e^{2iv\xi\sin^2(\theta/2)} - 1\right|\,|\theta|^{-1-\alpha}\,d\theta\,dv + 4\,(2/\pi)^\alpha\,\frac{\kappa}{E}. \end{aligned}$$

The successive changes of variables $u = \sin(\theta/2)$ and $y = \sqrt{|v\,\xi|}\,u$ imply that

$$\begin{aligned} \int_0^{\pi/2}\left|\,e^{2iv\xi\sin^2(\theta/2)} - 1\right|\,|\theta|^{-1-\alpha}\,d\theta &\leq 2^{1/2-\alpha}\int_0^{1/\sqrt{2}}\left|\,e^{2i|v\xi|u^2} - 1\right|\,|u|^{-1-\alpha}\,du \\ &\leq 2^{1/2-\alpha}\,|\xi|^{\alpha/2}\,|v|^{\alpha/2}\int_0^{\sqrt{|v\xi|/2}}\left|e^{2iy^2} - 1\right|\,|y|^{-1-\alpha}\,dy \\ &\leq 2^{1/2}\,|\xi|^{\alpha/2}\,|v|^{\alpha/2}\int_0^\infty\left|\,e^{2iy^2} - 1\right|\,|y|^{-1-\alpha}dy. \end{aligned}$$

□

Lemma 5.3.4 *Denote by M the constant given by Lemma 5.3.3. There exists some constant $C > 0$ such that, for $|\xi| \geq M$,*

$$|\hat{f}(\xi)| \leq C\,|\xi|^{-\alpha/2} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq \begin{cases} C\,|\xi|^{-\alpha/2}, & \alpha \leq 1, \\ C\,|\xi|^{\alpha/2-1}, & \alpha \geq 1. \end{cases} \tag{5.3.16}$$

Proof. We deduce from (5.3.13) that, for $\xi \geq M$,

$$\hat{f}(\xi) = \hat{f}(M)\, e^{-i\kappa(\xi-M)}\, e^{-\int_M^\xi \kappa \frac{q_1(u)}{u} du} + e^{-i\kappa\xi} \int_M^\xi e^{i\kappa\eta}\, e^{-\int_\eta^\xi \kappa \frac{q_1(u)}{u} du}\, \frac{q_2(\eta)}{\eta}\, d\eta.$$

Lemma 5.3.3 then implies that

$$|\hat{f}(\xi)| \leq C\, e^{-D_f \kappa |\xi|^\alpha/\alpha} + G_f\, e^{-D_f \kappa |\xi|^\alpha/\alpha} \int_M^{|\xi|} e^{D_f \kappa \eta^\alpha/\alpha}\, \eta^{\alpha/2-1}\, d\eta. \tag{5.3.17}$$

Similarly, we obtain the same estimate for $\xi \leq -M$. We now check that $|\xi|^{\alpha/2}\, |\hat{f}(\xi)|$ is bounded for $|\xi| \geq M$. By (5.3.17), it suffices to show that $F_1(\xi)/F_2(\xi)$ is bounded, where

$$F_1(\xi) = \int_M^{|\xi|} e^{D_f \kappa |\eta|^\alpha/\alpha}\, |\eta|^{\alpha/2-1}\, d\eta \qquad \text{and} \qquad F_2(\xi) = |\xi|^{-\alpha/2}\, e^{D_f \kappa |\xi|^\alpha/\alpha}.$$

Since $F_1'(\xi)/F_2'(\xi)$ tends to a constant as $|\xi| \to \infty$, we infer from l'Hospital's rule that $F_1(\xi)/F_2(\xi)$ tends to a constant as $|\xi| \to \infty$ and thus is bounded for $|\xi| \geq M$. This completes the proof of the first inequality of (5.3.16).

Proceeding as in the proof of (5.3.14), we have, for some $C > 0$,

$$|q_1(\xi)| \leq C\, |\xi|^\alpha, \qquad |\xi| \geq 1, \tag{5.3.18}$$

which together with equations (5.3.13), (5.3.15) and the first inequality of (5.3.16) implies that

$$|\hat{f}'(\xi)| \leq C\, (|\xi|^{-\alpha/2} + |\xi|^{\alpha/2-1}), \qquad |\xi| \geq M,$$

and, consequently, the second estimate of (5.3.16). □

Lemma 5.3.5 *Denote by M the constant given by Lemma 5.3.3. We assume that there exists $\delta \geq 0$ and $C > 0$ such that, for every $|\xi| \geq M$,*

$$|\hat{f}(\xi)| \leq C\, |\xi|^{-\delta} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C\, |\xi|^{-\delta}, \tag{5.3.19}$$

if $\alpha \in (0,1]$ and

$$|\hat{f}(\xi)| \leq C\, |\xi|^{-\delta} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C\, |\xi|^{-\delta+\alpha-1}, \tag{5.3.20}$$

if $\alpha \in (1,2)$. Then, for every $\varepsilon < \min\,(\alpha/2, 1-\alpha/2)$, there exists a constant $C(\varepsilon) > 0$ such that, for $|\xi| \geq M$,

$$|\hat{f}(\xi)| \leq C(\varepsilon)\, |\xi|^{-\delta-\alpha/2+\varepsilon} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C(\varepsilon)\, |\xi|^{-\delta-\alpha/2+\varepsilon}, \tag{5.3.21}$$

if $\alpha \in (0,1]$ and

$$|\hat{f}(\xi)| \leq C(\varepsilon)\, |\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C(\varepsilon)\, |\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)+\alpha-1}, \tag{5.3.22}$$

if $\alpha \in (1,2)$.

Proof. We first consider the case $\alpha \in (0,1]$. Let $\varepsilon > 0$ be such that $1-\alpha/2-\varepsilon > 0$ and $\varepsilon < \alpha/2$. Then,

$$\begin{aligned} |q_2(\xi)| \leq\ & \frac{\kappa}{E}\,|\xi|^{\alpha/2+\varepsilon} \int_{-\pi/4}^{\pi/4} \left|\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\right|^{1-\alpha/2-\varepsilon} \left|\int_0^1 \hat{f}'(\xi + u\xi(\cos\theta - 1))\,du\right|^{\alpha/2+\varepsilon} \\ & \times (1-\cos\theta)^{\alpha/2+\varepsilon}\,|\theta|^{-1-\alpha}\,d\theta + \frac{\kappa}{E}\int_{|\theta\pm\pi/4|\leq\pi/2} \hat{f}(\xi\sin\theta)\,|\theta|^{-1-\alpha}\,d\theta \\ & + \frac{\kappa}{E}\int_{3\pi/4\leq|\theta|\leq\pi} \left|\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\right|\,|\theta|^{-1-\alpha}\,d\theta. \end{aligned} \tag{5.3.23}$$

Since $\hat{f}$ satisfies (5.3.19), we thus deduce that

$$|q_2(\xi)| \leq C\,|\xi|^{-\delta+\alpha/2+\varepsilon} + C\,|\xi|^{-\delta} \leq C\,|\xi|^{-\delta+\alpha/2+\varepsilon}, \qquad |\xi| \geq M.$$

Similarly to the proof of the first inequality of (5.3.16), we obtain

$$\begin{aligned} |\hat{f}(\xi)| &\leq C\,e^{-\kappa D_f|\xi|^\alpha/\alpha} + C\,e^{-\kappa D_f|\xi|^\alpha/\alpha}\int_M^{|\xi|} e^{\kappa D_f\eta^\alpha/\alpha}\,\eta^{-\delta+\alpha/2+\varepsilon-1}d\eta, \\ &\leq C\,|\xi|^{-\delta-\alpha/2+\varepsilon}, \end{aligned}$$

for $|\xi| \geq M$. Consequently, we infer from (5.3.13), (5.3.18) and the above estimates that

$$|\hat{f}'(\xi)| \leq C\,|\xi|^{-\delta-\alpha/2+\varepsilon}, \qquad |\xi| \geq M,$$

which completes the proof of (5.3.21).

We now turn our attention to the case $\alpha \in (1,2)$. Let $\varepsilon > 0$ be such that $1-\alpha/2-\varepsilon > 0$. Equations (5.3.23) and (5.3.20) lead to

$$|q_2(\xi)| \leq C\,|\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)+\alpha},$$

and (5.3.22) follows as previously. □

Proof of Theorem 5.3.2. Let f be a weak solution to (5.1.5)-(5.1.2). It is sufficient to prove that the Fourier transform $\hat{f}$ of f satisfies, for every $s \geq 0$,

$$\hat{f}(\xi) \leq \frac{C(s)}{1+|\xi|^s}, \qquad \xi \in \mathbb{R}, \tag{5.3.24}$$

where $C(s)$ denotes some positive constant. By induction, it follows from Lemmas 5.3.4 and 5.3.5 that, for every $s \geq 0$, there exists a constant $C'(s) > 0$ such that

$$\hat{f}(\xi) \leq C'(s)\,|\xi|^{-s}, \qquad |\xi| \geq M.$$

Since $f \in L^1(\mathbb{R})$, $\hat{f}$ is bounded and (5.3.24) holds. □

5.3.3 Uniqueness

We now show the following theorem.

Theorem 5.3.6 *Assume that b satisfies (5.1.3). For all field strengths $E > 0$, there is at most one weak solution to (5.1.5)-(5.1.2) whose moments of any order are finite and such that $\int_{\mathbb{R}} f(v)\,dv = 1$.*

Let f be a weak solution to (5.1.5)-(5.1.2) such that any moments of f are finite and $\int_{\mathbb{R}} f(v)\,dv = 1$. Then, classical truncation argument ensures that, for every integer $m \geq 2$,

$$\left(m\,\zeta E + \int_{-\pi}^{\pi} (1-\cos^m\theta - \sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta\right)\int_{\mathbb{R}} v^m f(v)\,dv = m\,E\int_{\mathbb{R}} v^{m-1} f(v)\,dv$$
$$+\sum_{k=1}^{[(m-1)/2]}\binom{m}{2k}\int_{-\pi}^{\pi}(\cos\theta)^{m-2k}\,(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta\int_{\mathbb{R}} v^{2k} f(v)\,dv\int_{\mathbb{R}} v^{m-2k} f(v)\,dv, \tag{5.3.25}$$

where $\zeta = (\sqrt{K^2+4E^2}-K)/(2E)$. We set $w_m := \int_{\mathbb{R}} f(v)\,v^m\,dv/m!$, $m \in \mathbb{N}$. Then, we have

$$w_0 = 1, \qquad w_1 = \zeta \qquad \text{and} \qquad w_2 = \frac{1}{2}, \tag{5.3.26}$$

and the sequence $(w_m)_{m\in\mathbb{N}}$ satisfies

$$w_m = A_m\,w_{m-1} + \sum_{k=1}^{[(m-1)/2]} B_{m,k}\,w_{2k}\,w_{m-2k}, \qquad m \geq 3, \tag{5.3.27}$$

where

$$A_m = \frac{E}{m\,\zeta E + \int_{-\pi}^{\pi}(1-\cos^m\theta-\sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta}, \tag{5.3.28}$$

$$B_{m,k} = \frac{\int_{-\pi}^{\pi}(\cos\theta)^{m-2k}(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta}{m\,\zeta E + \int_{-\pi}^{\pi}(1-\cos^m\theta-\sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta}. \tag{5.3.29}$$

Lemma 5.3.7 *The coefficients $B_{m,k}$ defined by (5.3.29) satisfy*

$$B_{m,k} \leq \frac{2}{m-1}, \qquad 1 \leq k \leq [m/2]-1, \quad m \geq 3. \tag{5.3.30}$$

Proof. If $m = 2p$, $p \geq 2$, we have

$$\begin{aligned} 1-(\cos\theta)^{2p}-(\sin\theta)^{2p} &= (\cos^2\theta+\sin^2\theta)^p-(\cos\theta)^{2p}-(\sin\theta)^{2p} \\ &= \sum_{k=1}^{p-1}\binom{p}{k}(\cos\theta)^{2p-2k}(\sin\theta)^{2k}. \end{aligned}$$

We notice that $\binom{p}{k} \geq p$ for $1 \leq k \leq p-1$ and we thus deduce that

$$1 - (\cos\theta)^{2p} - (\sin\theta)^{2p} \geq p\,(\cos\theta)^{2p-2k}(\sin\theta)^{2k}, \qquad 1 \leq k \leq p-1,$$

whence $B_{2p,k} \leq 1/p$ and (5.3.30) holds when m is even.

For $m = 2p+1$, $B_{m,k}$ reads

$$B_{2p+1,k} = \frac{\int_{-\pi}^{\pi}(\cos\theta)^{2p+1-2k}(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta}{(2p+1)\,\zeta E + \int_{-\pi}^{\pi}(1-(\cos\theta)^{2p+1})\,|\theta|^{-1-\alpha}\,d\theta}.$$

We have

$$1 - (\cos\theta)^{2p+1} \geq 1 - (\cos\theta)^{2p} \geq 1 - (\cos\theta)^{2p} - (\sin\theta)^{2p}.$$

Consequently, for $1 \leq k \leq p-1$, we have

$$B_{2p+1,k} \leq \frac{\int_{-\pi}^{\pi}(\cos\theta)^{2p-2k}(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta}{2p\,\zeta E + \int_{-\pi}^{\pi}(1-(\cos\theta)^{2p}-(\sin\theta)^{2p})\,|\theta|^{-1-\alpha}\,d\theta} = B_{2p,k} \leq \frac{1}{p},$$

which completes the proof of (5.3.30). □

Lemma 5.3.8 *The coefficients A_m defined by (5.3.28) satisfy*

$$A_{2p} \leq \frac{1}{2}, \qquad p \geq 2. \tag{5.3.31}$$

Proof. It suffices to show that $A_4 \leq 1/2$. We claim that

$$\int_{-\pi}^{\pi}(1-\cos^4\theta-\sin^4\theta)\,|\theta|^{-1-\alpha}\,d\theta \geq \frac{3}{10}\int_{-\pi}^{\pi}(1-\cos\theta)\,|\theta|^{-1-\alpha}\,d\theta. \tag{5.3.32}$$

Then, we obtain

$$A_4 \leq \frac{E/K}{4\,\zeta E/K + 3/10},$$

where K is given by (5.3.11). Setting $x = E/K$, it is easily checked that

$$\frac{x}{2\sqrt{1+4x^2} - 17/10} \leq \frac{1}{2}, \qquad x \in \mathbb{R}_+,$$

and thus that $A_4 \leq 1/2$.

It now remains to prove (5.3.32). We point out that

$$1 - \cos^4\theta - \sin^4\theta = \frac{1}{2}\,\sin^2(2\,\theta), \qquad \theta \in [-\pi,\pi].$$

Consequently,

$$\int_{-\pi}^{\pi}(1-\cos^4\theta-\sin^4\theta)\,|\theta|^{-1-\alpha}\,d\theta = \int_0^{\pi}\theta^{-1-\alpha}\,\sin^2(2\theta)\,d\theta = 2^{\alpha}\int_0^{2\pi}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta.$$

Moreover, we have

$$\int_{-\pi}^{\pi} (1-\cos\theta)\,|\theta|^{-1-\alpha}\,d\theta = 4\int_0^{\pi} \theta^{-1-\alpha}\,\sin^2(\theta/2)\,d\theta = 2^{2-\alpha}\int_0^{\pi/2} \theta^{-1-\alpha}\,\sin^2\theta\,d\theta.$$

We then deduce that

$$\frac{\int_{-\pi}^{\pi}(1-\cos^4\theta-\sin^4\theta)\,|\theta|^{-1-\alpha}\,d\theta}{\int_{-\pi}^{\pi}(1-\cos\theta)\,|\theta|^{-1-\alpha}\,d\theta} = 4^{\alpha-1}\left(1+\frac{\int_{\pi/2}^{2\pi}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta}{\int_0^{\pi/2}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta}\right).$$

But,

$$\int_{\pi/2}^{2\pi}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta \geq (2\pi)^{-1-\alpha}\int_{\pi/2}^{2\pi}\sin^2\theta\,d\theta \geq (2\pi)^{-1-\alpha}\frac{3\pi}{4},$$

and, since $\alpha < 2$,

$$\int_0^{\pi/2}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta \leq \int_0^{\pi/2}\theta^{1-\alpha}\,d\theta \leq \left(\frac{\pi}{2}\right)^{2-\alpha}\frac{1}{2-\alpha}.$$

Thus,

$$\frac{\int_{\pi/2}^{2\pi}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta}{\int_0^{\pi/2}\theta^{-1-\alpha}\,\sin^2\theta\,d\theta} \geq (2-\alpha)\,4^{-\alpha}\,\frac{3}{2\pi^2},$$

and

$$\int_{-\pi}^{\pi}(1-\cos^4\theta-\sin^4\theta)\,|\theta|^{-1-\alpha}\,d\theta \geq \left(4^{\alpha-1}+(2-\alpha)\,\frac{3}{8\pi^2}\right)\int_{-\pi}^{\pi}(1-\cos\theta)\,|\theta|^{-1-\alpha}\,d\theta.$$

Since $\alpha \mapsto 4^{\alpha-1}+3\,(2-\alpha)/(8\pi^2)$ is a non-decreasing function, we obtain

$$\int_{-\pi}^{\pi}(1-\cos^4\theta-\sin^4\theta)\,|\theta|^{-1-\alpha}\,d\theta \geq \left(\frac{1}{4}+\frac{3}{4\pi^2}\right)\int_{-\pi}^{\pi}(1-\cos\theta)\,|\theta|^{-1-\alpha}\,d\theta.$$

Therefore, (5.3.32) holds. □

Lemma 5.3.9 *The coefficients A_m and $B_{m,k}$ defined by (5.3.28) and (5.3.29) satisfy*

$$A_{2p+1} + w_1\,B_{2p+1,p} \leq \frac{1}{2}+\frac{1}{2p}, \qquad p \geq 1. \tag{5.3.33}$$

Proof. We first consider the case $p = 1$. Since ζ satisfies $\zeta^2 + (K/E)\,\zeta - 1 = 0$, we have

$$A_3 + w_1\,B_{3,1} = \zeta + \frac{2K\zeta - 2E}{3\,\zeta E + \int_{-\pi}^{\pi}(1-\cos^3\theta)\,|\theta|^{-1-\alpha}\,d\theta}.$$

Making use of the inequalities

$$1-\cos^3\theta \geq \frac{1}{2}\,(1-\cos\theta), \qquad 1-\cos^3\theta \leq 3\,(1-\cos\theta), \qquad \theta\in[0,\pi],$$

we obtain

$$A_3 + w_1\, B_{3,1} \leq \zeta + \frac{2\,\zeta}{3\,\zeta E/K + 1/2} - \frac{2E/K}{3\,\zeta E/K + 3}.$$

Let us show that the right hand side of this inequality is less than 1. Setting $x = E/K$, this amounts to showing that

$$\frac{\sqrt{1+4x^2}-1}{2x} + \frac{2\,(\sqrt{1+4x^2}-1)}{3x\sqrt{1+4x^2}-2x} - \frac{4x}{3\,(\sqrt{1+4x^2}+1)} \leq 1, \qquad x \in \mathbb{R}_+,$$

that is,

$$\frac{2x\,(3\sqrt{1+4x^2}+10)}{3\,(3\,\sqrt{1+4x^2}-2)(\sqrt{1+4x^2}+1)} \leq 1, \qquad x \in \mathbb{R}_+.$$

Consequently, we want to show that

$$3\,(1-2x)\sqrt{1+4x^2} + 36x^2 - 20x + 3 \geq 0, \qquad x \in \mathbb{R}_+.$$

This holds for $x \in [0, 1/2]$ since $36x^2 - 20x + 3 \geq 0$. For $x \in (1/2, \infty)$, we use the inequality $\sqrt{1+4x^2} \leq 1 + 2x$. It then suffices to check that

$$12x^2 - 10x + 3 \geq 0, \qquad x \in (1/2, \infty),$$

which completes the proof of (5.3.33) for $p = 1$.

We now consider the general case $p \geq 2$. We have

$$\cos\theta\,(\sin\theta)^{2p} \leq 1 - \cos\theta, \qquad 1 - (\cos\theta)^{2p+1} \geq \frac{1}{2}\,(1-\cos\theta), \qquad \theta \in (0,\pi), \quad p \geq 2.$$

Consequently,

$$A_{2p+1} + w_1\, B_{2p+1,p} \leq \frac{E/K + \zeta}{(2p+1)\,\zeta E/K + 1/2}.$$

Proving that the right hand side of this inequality is bounded from above by $1/2 + 1/(2p)$ is equivalent to showing that

$$x + \frac{\sqrt{1+4x^2}-1}{2x} \leq \frac{p+1}{4p}\,((2p+1)\sqrt{1+4x^2} - 2p), \qquad x \in \mathbb{R}_+,$$

where x stands for E/K. This amounts to proving that

$$\frac{\sqrt{1+4x^2}-1}{x} + 2x + (p+1) - \left(p + \frac{3}{2} + \frac{1}{2p}\right)\sqrt{1+4x^2} \leq 0. \qquad x \in \mathbb{R}_+.$$

Since

$$\frac{\sqrt{1+4x^2}-1}{x} \leq 2x, \qquad x \in \mathbb{R}_+,$$

it suffices to check that $\varphi_p(x) \leq 0$ for $x \in \mathbb{R}_+$, $p \geq 2$, where

$$\varphi_p(x) = 4x + (p+1) - \left(p + \frac{3}{2} + \frac{1}{2p}\right)\sqrt{1+4x^2}, \qquad x \in \mathbb{R}_+, \quad p \geq 2.$$

For $x \geq 0$, we have

$$\varphi_p(x) \leq \varphi_p\left(\frac{1}{\sqrt{(p+3/2+1/(2p))^2-4}}\right) = p+1-\sqrt{\left(p+\frac{3}{2}+\frac{1}{2p}\right)^2-4}.$$

Let us check that

$$p+1-\sqrt{\left(p+\frac{3}{2}+\frac{1}{2p}\right)^2-4} \leq 0, \qquad p \geq 2. \tag{5.3.34}$$

For $p=2$, it holds. For $p \geq 3$, we have $p+1 \leq \sqrt{(p+3/2)^2-4}$, which implies (5.3.34).
□

Lemma 5.3.10 *The sequence $(w_m)_{m\in\mathbb{N}}$ defined by (5.3.26)-(5.3.27) is non-negative. Moreover, $(w_m)_{m\geq 2}$ is bounded from above by $1/2$.*

Proof. Let us first check that the coefficients $B_{m,k}$ are non-negative. This is straightforward when m is even. If m is odd, an integration by parts leads to

$$\int_{-\pi}^{\pi} (\cos\theta)^{m-2k}(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta = \frac{2\,(\alpha+1)}{2k+1}\int_0^{\pi}(\cos\theta)^{m-2k-1}(\sin\theta)^{2k+1}\,|\theta|^{-2-\alpha}\,d\theta$$
$$+\frac{m-2k-1}{2k+1}\int_{-\pi}^{\pi}(\cos\theta)^{m-2(k+1)}(\sin\theta)^{2(k+1)}\,|\theta|^{-1-\alpha}\,d\theta. \tag{5.3.35}$$

For $k=(m-1)/2$, we deduce that, for $m \geq 3$,

$$\int_{-\pi}^{\pi}\cos\theta(\sin\theta)^{m-1}\,|\theta|^{-1-\alpha}\,d\theta = \frac{2(\alpha+1)}{m}\int_0^{\pi}\sin^m\theta\,|\theta|^{-2-\alpha}\,d\theta,$$

is non-negative. Since the first integral in the right hand side of (5.3.35) is non-negative, it then follows by induction according to the decreasing values of k that (5.3.35) is non-negative for $1 \leq k \leq [(m-1)/2]$, $m \geq 3$. It then implies that the coefficients $B_{m,k}$ are non-negative and, then that w_m is non-negative for every $m \in \mathbb{N}$.

We infer from Lemmas 5.3.7, 5.3.8 and 5.3.9 that the sequence $(w_m)_{m\geq 2}$ is bounded by $1/2$. □

Proof of Theorem 5.3.6. Let us assume that there exist two weak solutions f and g to (5.1.5)-(5.1.2) whose moments of any order are finite and such that

$$\int_{\mathbb{R}} f(v)\,dv = \int_{\mathbb{R}} g(v)\,dv = 1.$$

Then the moments of f and g both satisfy (5.3.25). We then deduce by induction that

$$u_m := \int_{\mathbb{R}} f(v)\,v^m\,dv = \int_{\mathbb{R}} g(v)\,v^m\,dv, \qquad m \in \mathbb{N}.$$

Since f has finite moments of any order, $\hat{f} \in \mathcal{C}^\infty(\mathbb{R})$ and $|\hat{f}^{(m)}(\xi)| \leq \tilde{u}_m$, where $\tilde{u}_m := \int_{\mathbb{R}} f(v)\,|v|^m\,dv$, $m \in \mathbb{N}$. Let $r \in (0,1)$ and $t \in \mathbb{R}$. The Taylor Lagrange formula reads

$$\left|\hat{f}(t+h) - \sum_{m=0}^{k} \frac{\hat{f}^{(m)}(t)}{m!}\,h^m\right| \leq \frac{\tilde{u}_{k+1}}{(k+1)!}\,h^{k+1}, \qquad h \in \mathbb{R}, \quad k \in \mathbb{N}.$$

By Lemma 5.3.10, we know that $0 \leq u_m/m! \leq 1/2$ for $m \geq 2$. Thus, $\tilde{u}_{2k}/(2k)! \leq 1/2$, $k \geq 1$. Since $|v|^{2k-1} \leq 1 + |v|^{2k}$, we deduce that

$$\frac{\tilde{u}_{2k-1}}{(2k-1)!} \leq \frac{1}{(2k-1)!} + \frac{\tilde{u}_{2k}}{(2k-1)!} \leq \frac{1}{(2k-1)!} + k, \qquad k \geq 1.$$

Consequently, we have, for every $k \in \mathbb{N}$,

$$\frac{\tilde{u}_{k+1}}{(k+1)!} \leq \frac{k+4}{2},$$

and, for $|h| \leq r$,

$$\lim_{k\to+\infty} \frac{\tilde{u}_{k+1}}{(k+1)!}\,h^{k+1} = 0$$

holds. Therefore, for $|h| \leq r$,

$$\hat{f}(t+h) = \sum_{m=0}^{\infty} \frac{\hat{f}^{(m)}(t)}{m!}\,h^m.$$

The same argument gives

$$\hat{g}(t+h) = \sum_{m=0}^{\infty} \frac{\hat{g}^{(m)}(t)}{m!}\,h^m, \qquad |h| \leq r.$$

Since $\hat{f}^{(m)}(0) = (-i)^m u_m = \hat{g}^{(m)}(0)$, we deduce that $\hat{f} = \hat{g}$ on $(-r, r)$. By bootstrap, we obtain that $\hat{f} = \hat{g}$ on $\mathbb{R}$. The proof of Theorem 5.3.2 implies that $\hat{f}$ and $\hat{g}$ both belong to $L^1(\mathbb{R})$ and the Fourier inversion theorem gives $f = g$. □

5.4 Numerical experiments

In this section we show the results from some numerical experiments, that were carried out to illustrate the theoretical results. We solved equation (5.3.13) as an initial value problem starting at $\xi = 0$ with $\hat{f} = 1$. The equations were solved iteratively by computing first q_1 and q_2 by standard quadrature routines, and then integrating the ordinary differential equation by a Runge-Kutta method. Then finally the distribution $f(v)$ was obtained by Fast Fourier Transform. All was done using standard routines in Matlab in a relatively straight forward way.

Figure 5.1 shows the Fourier transform $\hat{f}$ for $\alpha = 1.0$ and $E = 3.0$; obviously, because f is real, $\Re\hat{f}$ is even and $\Im\hat{f}$ is odd. Then Figure 5.2 shows the result of computing the inverse transform for obtaining the function f. Also here $E = 3.0$, but the results for three different values of α are shown.

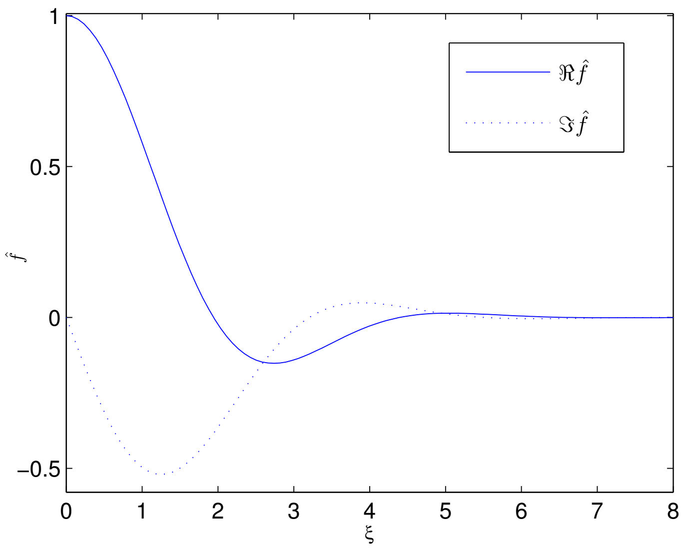

Figure 5.1: The Fourier transform of $f(v)$ for $E = 3.0$ and $b(\theta) = |\theta|^{-1-\alpha}$ with $\alpha = 1.0$.

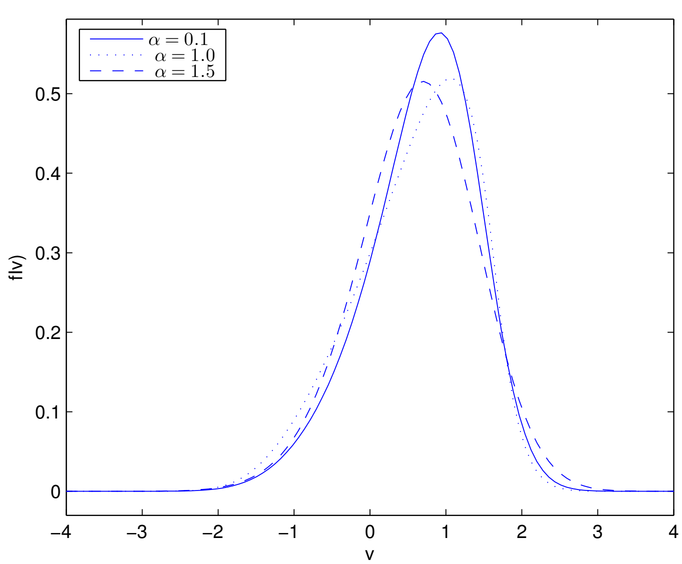

Figure 5.2: The solution $f(v)$ for $E = 3.0$ and $b(\theta) = |\theta|^{-1-\alpha}$ with $\alpha = 0.1, 1.0$ and 1.5.

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Partie IV

Equation de coagulation de Oort-Hulst-Safronov

Cette quatrième partie, composée d'un chapitre, est consacrée à l'étude de l'équation de coagulation de Oort-Hulst-Safronov, introduite dans la Section 1.2.4. Cette équation est approchée par une suite d'équations discrètes. Ce travail a fait l'objet d'un article publié dans le journal *Mathematical Methods in the Applied Sciences.*

Chapitre 6

Convergence of a discrete version of the Oort-Hulst-Safronov coagulation equation

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Abstract

A discrete version of the Oort-Hulst-Safronov (OHS) coagulation equation is studied. Besides the existence of a solution to the Cauchy problem, it is shown that solutions to a suitable sequence of those discrete equations converge towards a solution to the OHS equation.

6.1 Introduction

We study here a discrete approximation to the Oort-Hulst-Safronov (OHS) coagulation equation which describes the growth by accretion of stellar objects. More generally, coagulation models aim at describing the process by which particles encounter and merge into a single one, each particle being fully identified by a size variable (volume, mass, length, ...). The evolution of these particles is then described by a size distribution function $f(t,x) \geq 0$ which represents the density of particles of size $x \in (0,+\infty)$ at time $t \geq 0$. Besides the classical Smoluchowski coagulation equation originally introduced in colloidal chemistry (see [1, 10] and the references therein), a different coagulation equation has been derived independently in an astrophysical context by Oort and van de Hulst [12] and reads [13]

$$\begin{aligned} \partial_t f &= Q(f), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}_+, && (6.1.1)\\ f(0,x) &= f^{in}(x), \qquad x \in \mathbb{R}_+, && (6.1.2) \end{aligned}$$

where

$$Q(f) = -\partial_x \left[f(t,x) \int_0^x y\, K(x,y) f(t,y)\, dy \right] - \int_x^\infty K(x,y) f(t,x) f(t,y)\, dy. \qquad (6.1.3)$$

Here, ∂_t and ∂_x denote the partial derivatives with respect to time t and size x, respectively, and the coagulation kernel K is a non-negative and symmetric function accounting for the physics of the coalescence process. Approximating ∂_x by an upwind finite difference scheme and the integrals by Riemann sums, one obtains a discrete version of (6.1.1) which reads

$$\begin{aligned} \frac{dc_i}{dt} &= Q_i(c) \qquad \text{in} \quad (0,+\infty), && (6.1.4)\\ c_i(0) &= c_i^{in}, && (6.1.5) \end{aligned}$$

for $i \geq 1$, where $c = (c_i)_{i\geq 1}$,

$$Q_i(c) = c_{i-1} \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j - c_i \sum_{j=1}^{i} j\, K_{i,j}\, c_j - \sum_{j=i}^{\infty} K_{i,j}\, c_i\, c_j, \qquad (6.1.6)$$

and $K_{i,j} = K_{j,i} \geq 0$ is the discrete coagulation kernel. Equations (6.1.4)-(6.1.6) are also a particular case of a two-parameter family of discrete coagulation models introduced by Dubovski [5], where a link with the OHS equation is highlighted. In the sequel, equations (6.1.4)-(6.1.6) will be referred to as the discrete OHS (dOHS) equation.

Our aim is here to justify the connection between the OHS and dOHS equations. A similar relationship has been established in [9] between the classical continuous and discrete coagulation-fragmentation equations. We adapt herein the approach developed there. Roughly speaking, the main idea is to realize that the dOHS equation may be seen as a modified OHS equation. More precisely, let $c = (c_i)_{i\geq 1}$ be a solution to (6.1.4)-(6.1.6) and $(\varphi_i)_{i\geq 1}$ a sequence of (sufficiently rapidly decaying) real numbers. Then, thanks to the symmetry property $K_{i,j} = K_{j,i}$, a weak formulation for (6.1.4)-(6.1.6) reads

$$\frac{d}{dt}\sum_{i=1}^{\infty} c_i\,\varphi_i = \sum_{i=1}^{\infty}\sum_{j=1}^{i} K_{i,j}\,c_i\,c_j\,[\,j\,(\varphi_{i+1}-\varphi_i)-\varphi_j]. \tag{6.1.7}$$

We show that (6.1.7) may be interpreted as a weak formulation of a modified OHS equation. To this end, we fix $\varepsilon \in (0,1)$ and set

$$\Lambda_i^\varepsilon = [(i-1/2)\varepsilon,(i+1/2)\varepsilon) \qquad \text{and} \qquad \chi_i^\varepsilon = \mathbf{1}_{\Lambda_i^\varepsilon}, \tag{6.1.8}$$

for $i \geq 1$. We next introduce, for $(t,x,y) \in \mathbb{R}_+^3$,

$$f_\varepsilon(t,x) = \sum_{i=1}^{\infty} c_i(t)\,\chi_i^\varepsilon(x) \qquad \text{and} \qquad K_\varepsilon(x,y) = \sum_{i,j=1}^{\infty}\frac{K_{i,j}}{\varepsilon}\,\chi_i^\varepsilon(x)\,\chi_j^\varepsilon(y).$$

For $\varphi \in \mathcal{D}(\mathbb{R}_+)$, we define the approximated ε-step function of φ by

$$\varphi_\varepsilon(x) = \sum_{i=1}^{\infty}\varphi_i^\varepsilon\,\chi_i^\varepsilon(x) \qquad \text{with} \qquad \varphi_i^\varepsilon = \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon}\varphi(y)\,dy, \tag{6.1.9}$$

for every $x \in \mathbb{R}_+$. Finally, for any function g from $\mathbb{R}_+$ to $\mathbb{R}$ of the form

$$g(x) = \sum_{i=1}^{\infty} g_i\,\chi_i^\varepsilon(x), \qquad g_i \in \mathbb{R},$$

we define the discrete size derivative $D_\varepsilon(g)$ of g by

$$D_\varepsilon(g)(x) = \frac{1}{\varepsilon}\sum_{i=1}^{\infty}(g_{i+1}-g_i)\,\chi_i^\varepsilon(x), \qquad x \in \mathbb{R}_+.$$

As we shall see in Lemma 6.3.6 below, for $\varphi \in \mathcal{D}(\mathbb{R}_+)$, $(\varphi_\varepsilon, D_\varepsilon(\varphi_\varepsilon))$ converges a.e. towards $(\varphi, \partial_x\varphi)$, the function φ_ε being defined by (6.1.9).

With the previous notations, (6.1.7) reads

$$\frac{d}{dt}\left(\int_0^\infty f_\varepsilon\,\varphi_\varepsilon\,dx\right) = \int_0^\infty\int_0^{r_\varepsilon(x)} K_\varepsilon(x,y)\,f_\varepsilon(t,x)\,f_\varepsilon(t,y)\,[\,y\,D_\varepsilon(\varphi_\varepsilon)(x)-\varphi_\varepsilon(y)]\,dy\,dx, \tag{6.1.10}$$

for $\varphi \in \mathcal{D}(\mathbb{R}_+)$, where

$$r_\varepsilon(x) = \left(\left[\frac{x}{\varepsilon}+\frac{1}{2}\right]+\frac{1}{2}\right)\varepsilon, \tag{6.1.11}$$

denoting by $[u]$ the integer part of the real number u.

Thus, if we suppose that (f_ε) converges towards some function f and if (K_ε) converges towards some K, then we may pass formally to the limit in (6.1.10). Thereby, we obtain that f satisfies, for every $\varphi \in \mathcal{D}(\mathbb{R}_+)$,

$$\frac{d}{dt}\int_0^\infty f\,\varphi\,dx = \int_0^\infty \int_0^x K(x,y)f(t,x)f(t,y)\,[\,y\,\partial_x\varphi(x) - \varphi(y)]\,dy\,dx,$$

which turns out to be the weak formulation of the OHS equation (6.1.1) (see (6.2.6) below). Observe that the convergence of K_ε to a finite limit requires that $K_{i,j}$ depends on ε.

We now describe the contents of the paper. We first introduce a sequence of approximated discrete equations of the OHS equation and state our main results in the next section. We then show, in Section 6.3, the convergence of a sequence of solutions to these discrete models towards a solution to the OHS model. For the sake of completeness, the Cauchy problem for the dOHS equation is investigated in Section 6.4. We finally illustrate the convergence theorem by a numerical comparison between an explicit solution and the associated discrete solution.

6.2 Main results

We make here the same assumptions as in [6]: we require that the classical symmetry condition is fulfilled, namely

$$0 \le K(x,y) = K(y,x), \qquad (x,y) \in \mathbb{R}_+^2, \tag{6.2.1}$$

and that

$$K \in W^{1,\infty}_{loc}([0,+\infty)^2), \tag{6.2.2}$$
$$\partial_x K(x,y) \ge -\alpha, \qquad \text{for some } \alpha \ge 0. \tag{6.2.3}$$

We also suppose that K is *strictly subquadratic*, that is, for each $R \ge 1$,

$$\omega_R(y) = \sup_{x\in[0,R]} \frac{K(x,y)}{y} \longrightarrow 0 \quad \text{as} \quad y \to +\infty. \tag{6.2.4}$$

Concerning the initial condition, we assume that

$$f^{in} \in L^1_1(\mathbb{R}_+) = L^1(\mathbb{R}_+,(1+x)dx) \qquad \text{and} \qquad f^{in} \ge 0 \quad a.e. \tag{6.2.5}$$

The notion of weak solutions to the OHS equation we consider here is the same as in [6] and is as follows:

Definition 6.2.1 *Assume that K satisfies (6.2.1)-(6.2.4) and that f^{in} satisfies (6.2.5). A function $f = f(t,x)$ is said to be a weak solution to the OHS equation (6.1.1)-(6.1.3) with initial condition f^{in} if*

$$0 \;\le\; f \in \mathcal{C}([0,T]; w - L^1(\mathbb{R}_+)) \;\cap\; L^\infty(0,T; L^1_1(\mathbb{R}_+)) \qquad \textit{for every } T \in \mathbb{R}_+,$$

and, for all $\varphi \in \mathcal{D}(\mathbb{R}_+)$ *and* $t > 0$,

$$\int_0^\infty f(t,x)\,\varphi(x)\,dx - \int_0^\infty f^{in}(x)\,\varphi(x)\,dx$$
$$= \int_0^t \int_0^\infty \int_0^x K(x,y)\,f(s,x)\,f(s,y)\,[\,y\,\partial_x\varphi(x) - \varphi(y)]\,dy\,dx\,ds. \quad (6.2.6)$$

Here $\mathcal{C}([0,T]; w - L^1(\mathbb{R}_+))$ denotes the space of weakly continuous functions in $L^1(\mathbb{R}_+)$, that is the space of continuous functions from $[0,T]$ in $L^1(\mathbb{R}_+)$ endowed with its weak topology. Recall that it follows from [6, Theorem 2.2] that there exists at least a weak solution to the OHS equation (6.1.1)-(6.1.3) in the sense of Definition 6.2.1 when K and f^{in} fulfil (6.2.1)-(6.2.4) and (6.2.5), respectively.

We now introduce the discrete approximations to the OHS equation. We fix $\varepsilon \in (0,1)$ and define discrete coefficients $K^\varepsilon_{i,j}$ either by

$$K^\varepsilon_{i,j} = \frac{1}{\varepsilon}\int_{\Lambda^\varepsilon_i \times \Lambda^\varepsilon_j} K(x,y)\,dy\,dx, \quad (6.2.7)$$

or by

$$K^\varepsilon_{i,j} = \varepsilon K(\varepsilon i, \varepsilon j), \quad (6.2.8)$$

for $i,j \geq 1$. In both case, (6.2.1) and (6.2.4) imply that the following properties hold:

$$0 \leq K^\varepsilon_{i,j} = K^\varepsilon_{j,i}, \qquad i,j \geq 1 \quad (6.2.9)$$

$$\lim_{j\to\infty} \frac{K^\varepsilon_{i,j}}{j} = 0 \qquad \text{for each } i \geq 1. \quad (6.2.10)$$

We next define the discrete initial condition $c^{in,\varepsilon} = (c^{in,\varepsilon}_i)_{i\geq 1}$ by

$$c^{in,\varepsilon}_i = \frac{1}{\varepsilon}\int_{\Lambda^\varepsilon_i} f^{in}(x)\,dx, \qquad i \geq 1. \quad (6.2.11)$$

It is straightforward to check that

$$\varepsilon \sum_{i=1}^\infty c^{in,\varepsilon}_i \leq \int_0^\infty f^{in}(x)\,dx, \quad (6.2.12)$$

and

$$\varepsilon^2 \sum_{i=1}^\infty i\,c^{in,\varepsilon}_i \leq 2\int_0^\infty x\,f^{in}(x)\,dx. \quad (6.2.13)$$

Let $c^\varepsilon = (c^\varepsilon_i)_{i\geq 1}$ be a solution to the dOHS equation (in the sense of Definition 6.2.3 below) with the coefficients $K^\varepsilon_{i,j}$ and the initial condition $c^{in,\varepsilon}$ such that

$$\sum_{i=1}^\infty i\,c^\varepsilon_i(t) \leq \sum_{i=1}^\infty i\,c^{in,\varepsilon}_i, \qquad t \geq 0, \quad (6.2.14)$$

(see Section 6.4 for the existence of such a solution).

Similarly to what was done in Section 6.1, we introduce continuous formulations for the discrete quantities and set

$$f_\varepsilon(t,x) = \sum_{i=1}^\infty c_i^\varepsilon(t)\,\chi_i^\varepsilon(x), \tag{6.2.15}$$

$$K_\varepsilon(x,y) = \sum_{i,j=1}^\infty \frac{K_{i,j}^\varepsilon}{\varepsilon}\,\chi_i^\varepsilon(x)\,\chi_j^\varepsilon(y), \tag{6.2.16}$$

for $(t,x,y)\in\mathbb{R}_+^3$. With these notations, K_ε converges a.e. towards K and satisfies (6.2.1).

Our main results are the following.

Theorem 6.2.2 *Assume that K satisfies (6.2.1)-(6.2.4) and that f^{in} satisfies (6.2.5). We denote by c^ε a solution to the dOHS equation (6.1.4)-(6.1.6) with the coefficient $K_{i,j}^\varepsilon$ defined by (6.2.7) or (6.2.8) and with the initial data $c^{in,\varepsilon}$ defined by (6.2.11) such that (6.2.14) holds. Let f_ε be the function defined by (6.2.15). Then there exist a weak solution f to the OHS equation (6.1.1)-(6.1.3) with initial data f^{in} and a subsequence (f_{ε_n}) of (f_ε) such that*

$$f_{\varepsilon_n} \longrightarrow f \quad \text{in } \mathcal{C}([0,T]; w-L^1(\mathbb{R}_+)) \quad \text{for each } T\in\mathbb{R}_+.$$

As a by-product of Theorem 6.2.2, we obtain the existence of a weak solution to the OHS equation, thus providing, under the same set of assumptions, an alternative proof of [6, Theorem 2.2]. The existence proof in [6] also relies on weak compactness but with a different approximation scheme. Let us also point out that the approximation to the OHS equation developed in this paper might be used for numerical simulations, as illustrated in Section 6.5.

We also show the existence of solutions to the dOHS equation, arguing as in [3, 14]. Let us first give the definition of a weak solution to the dOHS equation.

Definition 6.2.3 *Let $T\in(0,+\infty)$ and assume that $c^{in}=(c_i^{in})_{i\ge1}$ is a sequence of non-negative real numbers. A solution $c=(c_i)_{i\ge1}$ to the dOHS equation (6.1.4)-(6.1.6) on $[0,T)$ is a sequence of non-negative continuous functions such that, for each $i\ge1$ and $t\in(0,T)$,*

$$(i)\quad c_i\in\mathcal{C}([0,T)),\quad \sum_{j=i}^\infty K_{i,j}\,c_i\,c_j\in L^1(0,t),$$

$$(ii)\quad c_i(t)=c_i^{in}+\int_0^t\left[c_{i-1}\sum_{j=1}^{i-1} j\,K_{i-1,j}\,c_j - c_i\sum_{j=1}^{i} j\,K_{i,j}\,c_j-\sum_{j=i}^\infty K_{i,j}\,c_i\,c_j\right]ds.$$

The existence result for the dOHS equation then reads:

Proposition 6.2.4 *Let $(K_{i,j})$ be a sequence of non-negative real numbers such that*

$$K_{i,j} = K_{j,i} \geq 0 \qquad \text{and} \qquad \lim_{k\to+\infty} \frac{K_{i,k}}{k} = 0, \qquad i,j \geq 1. \tag{6.2.17}$$

If $c^{in} = (c_i^{in})$ is a sequence of non-negative real numbers such that

$$\sum_{i=1}^{\infty} i\, c_i^{in} < +\infty, \tag{6.2.18}$$

then there exists at least a solution c to the dOHS equation (6.1.4)-(6.1.6) on $[0, +\infty)$ such that (6.2.14) holds.

We also note that, unlike the OHS equation, the dOHS equation propagates perturbations with an infinite speed. More precisely, it follows from [6, Theorem 2.6] that, if f^{in} is compactly supported in $[0, +\infty)$, then $f(t)$ is also compactly supported for $t \in [0, T_*)$, where T_* might be finite or infinite according to the growth of the coagulation kernel K. On the opposite, the following proposition holds for the dOHS equation.

Proposition 6.2.5 *Assume that $K_{i,i} > 0$ for $i \geq 1$. Let $c^{in} = (c_i^{in})_{i\geq1}$ be a sequence of non-negative real numbers such that $c_k^{in} > 0$ for some $k \geq 1$ and $c = (c_i)_{i\geq1}$ be a solution to the dOHS equation (6.1.4)-(6.1.6) on some interval $[0, T)$ with initial condition c^{in}. Then, for all $i \geq k$ and $t \in (0, T)$, $c_i(t) > 0$.*

6.3 Proof of Theorem 6.2.2

We consider here the dOHS equation (6.1.4)-(6.1.6) with the coefficient $K_{i,j}^{\varepsilon}$ defined by (6.2.7) or (6.2.8) and with the initial data $c^{in,\varepsilon}$ defined by (6.2.11). The proof is performed in two steps: the main idea relies on L^1 weak compactness. We thus need uniform estimates with respect to ε for the function f_ε defined by (6.2.15), which corresponds to the first step. These estimates ensure that (f_ε) lies in a weakly compact set of L^1. In a second step, we pass to the limit as $\varepsilon \to 0$.

6.3.1 *A priori* estimates

We set

$$M = \int_0^\infty f^{in}(x)\,(1+x)\,dx, \tag{6.3.1}$$

and notice that, by (6.2.15),

$$\int_0^\infty f_\varepsilon(t,x)\,dx = \varepsilon \sum_{i=1}^{\infty} c_i^\varepsilon(t) \qquad \text{and} \qquad \int_0^\infty f_\varepsilon(t,x)\,x\,dx = \varepsilon^2 \sum_{i=1}^{\infty} i\, c_i^\varepsilon(t), \tag{6.3.2}$$

for every $t \geq 0$.

Lemma 6.3.1 *For all $t \geq 0$ and $\varepsilon \in (0,1)$, there holds*

$$\int_0^\infty f_\varepsilon(t,x)\, x\, dx \leq 2M. \tag{6.3.3}$$

Proof. Using successively (6.3.2), (6.2.14) and (6.2.13), we obtain that

$$\int_0^\infty f_\varepsilon(t,x)\, x\, dx = \varepsilon^2 \sum_{i=1}^\infty i\, c_i^\varepsilon(t) \leq \varepsilon^2 \sum_{i=1}^\infty i\, c_i^{in,\varepsilon} \leq 2 \int_0^\infty f^{in}(x)\, x\, dx,$$

whence (6.3.3). □

Lemma 6.3.2 *For all $t \geq 0$ and $\varepsilon \in (0,1)$, we have*

$$\int_0^\infty f_\varepsilon(t,x)\, dx \leq M. \tag{6.3.4}$$

Proof. Let $m \geq 1$. Taking $\varphi_i = i$ for $i \leq m$ and $\varphi_i = 0$ for $i > m$ in (6.1.7), we deduce from the non-negativity of $K_{i,j}^\varepsilon$ and c^ε that $\sum_{i=1}^m c_i^\varepsilon$ is a non-increasing function of time. Thus, for every $t \geq 0$,

$$\sum_{i=1}^m \varepsilon c_i^\varepsilon(t) \leq \sum_{i=1}^m \varepsilon c_i^{in,\varepsilon},$$

whence, by (6.2.12),

$$\sum_{i=1}^m \varepsilon c_i^\varepsilon(t) \leq \int_0^\infty f^{in}(x)\, dx. \tag{6.3.5}$$

We now let $m \to +\infty$ and deduce (6.3.4) thanks to (6.3.2). □

Lemma 6.3.3 *Let $\phi \in \mathcal{C}^2([0,+\infty))$ be a non-negative convex function such that $\phi(0) = 0$, $\phi'(0) = 1$ and ϕ' is concave. If*

$$L_\phi := \int_0^\infty \phi(f^{in})(x)\, dx < +\infty, \tag{6.3.6}$$

then, for every $T \in \mathbb{R}_+$, there exists a constant $C(T)$ such that, for all $t \in [0,T]$ and $\varepsilon \in (0,1)$, we have

$$\int_0^\infty \phi(f_\varepsilon(t,x))\, dx \leq C(T)\, L_\phi. \tag{6.3.7}$$

Proof. The concavity of ϕ', the non-negativity of $\phi'(0)$ and $\phi(0) = 0$ ensure that, for every $v \in \mathbb{R}_+$,

$$v\phi'(v) \leq 2\,\phi(v), \tag{6.3.8}$$

(see [7, Lemma A.1] for a proof).

Let $T > 0$, $R > 0$ and $m \in \mathbb{N}$ such that $R \in \Lambda_m^\varepsilon$. We infer from (6.1.7) and the non-negativity of $K_{i,j}^\varepsilon$, c^ε and ϕ' that

$$\frac{d}{dt}\sum_{i=1}^{m}\phi(c_i^\varepsilon) \leq \sum_{i=1}^{m-1}\sum_{j=1}^{i} j\, c_i^\varepsilon\, c_j^\varepsilon\, K_{i,j}^\varepsilon\,[\phi'(c_{i+1}^\varepsilon)-\phi'(c_i^\varepsilon)] - \sum_{j=1}^{m} j\, c_m^\varepsilon\, c_j^\varepsilon\, K_{m,j}^\varepsilon\,\phi'(c_m^\varepsilon). \tag{6.3.9}$$

Introducing

$$\psi(x) := x\,\phi'(x) - \phi(x), \qquad x \in \mathbb{R}_+,$$

it easily follows from (6.3.8), the non-negativity and the convexity of ϕ that ψ satisfies the following properties:

$$0 \leq \psi(x) \leq x\,\phi'(x) \qquad \text{and} \qquad \psi(x) \leq \phi(x), \qquad x \in \mathbb{R}_+. \tag{6.3.10}$$

Due to the convexity of ϕ, $\phi(x)-\phi(y) \geq (x-y)\phi'(y)$ for all $x, y \in \mathbb{R}_+$, and thus,

$$x(\phi'(y)-\phi'(x)) \leq \psi(y)-\psi(x), \qquad x, y \in \mathbb{R}_+. \tag{6.3.11}$$

Using (6.3.10) and (6.3.11), we deduce from (6.3.9) that

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{m}\phi(c_i^\varepsilon) &\leq \sum_{i=1}^{m-1}\sum_{j=1}^{i} j\, c_j^\varepsilon\, K_{i,j}^\varepsilon\,[\,\psi(c_{i+1}^\varepsilon)-\psi(c_i^\varepsilon)\,] - \sum_{j=1}^{m} j\, c_j^\varepsilon\, K_{m,j}^\varepsilon\,\psi(c_m^\varepsilon)\\ &\leq \sum_{i=2}^{m}\sum_{j=1}^{i-1} j\, c_j^\varepsilon\,[\,K_{i-1,j}^\varepsilon - K_{i,j}^\varepsilon\,]\,\psi(c_i^\varepsilon).\end{aligned}$$

We infer from the definition of $K_{i,j}^\varepsilon$ by (6.2.7) or (6.2.8) and from (6.2.3) that

$$K_{i-1,j}^\varepsilon - K_{i,j}^\varepsilon \leq \alpha\varepsilon^2 \qquad \text{for all } i \geq 2 \text{ and } j \geq 1.$$

Consequently, we obtain that

$$\frac{d}{dt}\sum_{i=1}^{m}\varepsilon\,\phi(c_i^\varepsilon) \leq \alpha\left(\sum_{i=1}^{m}\varepsilon\,\psi(c_i^\varepsilon)\right)\left(\sum_{j=1}^{\infty}\varepsilon^2\, j\, c_j^\varepsilon\right).$$

By (6.3.10), (6.2.14) and (6.2.13), we have

$$\frac{d}{dt}\sum_{i=1}^{m}\varepsilon\,\phi(c_i^\varepsilon) \leq 2\alpha M\sum_{i=1}^{m}\varepsilon\,\phi(c_i^\varepsilon).$$

Then, the successive use of the Gronwall Lemma and the Jensen inequality yields

$$\sum_{i=1}^{m}\varepsilon\,\phi(c_i^\varepsilon(t)) \leq C(T)\sum_{i=1}^{m}\varepsilon\,\phi(c_i^{in,\varepsilon}) \leq C(T)\sum_{i=1}^{m}\int_{\Lambda_i^\varepsilon}\phi(f^{in}(x))dx \leq C(T)L_\phi,$$

for every $t \in [0,T]$. Finally, since $(m+1/2)\varepsilon > R$, we deduce that

$$\int_0^R \phi(f_\varepsilon(t,x))\,dx \leq \varepsilon\sum_{i=1}^{m}\phi(c_i^\varepsilon(t)) \leq C(T)\,L_\phi, \qquad t \in [0,T].$$

Letting now $R \to +\infty$ completes the proof of Lemma 6.3.3. □

Lemma 6.3.4 *For all $T \in \mathbb{R}_+$ and $\psi \in \mathcal{C}_c^1([0,+\infty))$,*

$$t \longmapsto \int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx \quad \text{is bounded in} \quad W^{1,\infty}(0,T). \tag{6.3.12}$$

Proof. Let $\psi \in \mathcal{C}_c^1([0,+\infty))$ such that $\mathrm{supp}(\psi) \subset [0,R]$ and set

$$\psi_i^\varepsilon = \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon} \psi(x)\,dx, \qquad i \geq 1.$$

Denoting by m the integer such that $R \in \Lambda_m^\varepsilon$, we infer from (6.1.7) that

$$\begin{aligned} \left|\frac{d}{dt}\int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx\right| &= \varepsilon \left|\frac{d}{dt}\sum_{i=1}^\infty c_i^\varepsilon\,\psi_i^\varepsilon\right| \\ &= \varepsilon \left|\sum_{i=1}^m \sum_{j=1}^i j\,c_i^\varepsilon\,c_j^\varepsilon\,K_{i,j}^\varepsilon\,(\psi_{i+1}^\varepsilon - \psi_i^\varepsilon) - \sum_{i=1}^m \sum_{j=i}^\infty \psi_i^\varepsilon\,K_{i,j}^\varepsilon\,c_i^\varepsilon\,c_j^\varepsilon\right|. \end{aligned}$$

Since

$$\left|\frac{\psi_{i+1}^\varepsilon - \psi_i^\varepsilon}{\varepsilon}\right| \leq \|\psi\|_{W^{1,\infty}},$$

it follows from (6.2.13), (6.2.14) and (6.3.5) that

$$\left|\frac{d}{dt}\int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx\right| \leq M^2\left[3\,\|K\|_{L^\infty((0,R+1)^2)} + 2\sup_{j\geq m+1}\sup_{i\leq m}\frac{K_{i,j}^\varepsilon}{j\varepsilon^2}\right]\|\psi\|_{W^{1,\infty}}.$$

By (6.2.4) and (6.2.16), there exists, for each $R \geq 1$, a bounded non-increasing function $\overline{\omega}_R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\sup_{y\geq M}\sup_{x\in(0,R)}\frac{K_\varepsilon(x,y)}{y} \leq \overline{\omega}_R(M), \quad \text{for } M>0 \qquad \text{and} \qquad \lim_{M\to+\infty}\overline{\omega}_R(M) = 0. \tag{6.3.13}$$

Thus,

$$\left|\frac{d}{dt}\int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx\right| \leq M^2\left[3\|K\|_{L^\infty((0,R+1)^2)} + 4\sup_{y\geq R}\overline{\omega}_{R+1}(y)\right]\|\psi\|_{W^{1,\infty}},$$

which, together with Lemma 6.3.2, yields (6.3.12). □

6.3.2 Convergence

Lemma 6.3.5 *There exist a non-negative function f and a subsequence of (f_ε) (not relabelled) such that, for every $T \in (0,+\infty)$,*

$$f \in L^\infty(0,T;L_1^1(\mathbb{R}_+)) \qquad \text{and} \qquad f_\varepsilon \longrightarrow f \quad \text{in} \quad \mathcal{C}([0,T];w-L^1(\mathbb{R}_+)). \tag{6.3.14}$$

Proof. Let $T > 0$. Due to [15, Theorem 1.3.2], it suffices to check that

$$\text{the family } (f_\varepsilon) : [0,T] \to L^1(\mathbb{R}_+) \text{ is weakly equicontinuous,} \tag{6.3.15}$$
$$\text{the set } \{f_\varepsilon(t), \varepsilon \in (0,1)\} \text{ is weakly relatively compact in } L^1(\mathbb{R}_+), \tag{6.3.16}$$

for every $t \in [0,T]$, to conclude that (f_ε) is relatively sequentially compact in $\mathcal{C}([0,T]; w-L^1(\mathbb{R}_+))$.

We first prove (6.3.16). Since $f^{in} \in L^1(\mathbb{R}_+)$, a refined version of the de la Vallée Poussin theorem [4, 11] ensures the existence of a function ϕ fulfilling the assumptions of Lemma 6.3.3 and such that

$$\lim_{r \to +\infty} \frac{\phi(r)}{r} = 0 \qquad \text{and} \qquad \int_0^\infty \phi(f^{in})(x)\,dx < +\infty.$$

We then infer from Lemmas 6.3.1, 6.3.2 and 6.3.3 that

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \left\{ \int_0^\infty f_\varepsilon(t,x)\,(1+x)\,dx + \int_0^\infty \phi(f_\varepsilon(t,x))\,dx \right\} < +\infty, \tag{6.3.17}$$

whence (6.3.16) by the Dunford-Pettis theorem.

We now turn our attention to (6.3.15). Let $\varphi \in L^\infty(\mathbb{R}_+)$. There exists a sequence of functions (φ_k) in $\mathcal{C}_c^1(\mathbb{R}_+)$ such that

$$\varphi_k \longrightarrow \varphi \quad a.e. \text{ in } \mathbb{R}_+, \tag{6.3.18}$$
$$\|\varphi_k\|_{L^\infty} \leq \|\varphi\|_{L^\infty} \tag{6.3.19}$$

We fix $\eta \in (0,1)$. From (6.3.17), we deduce the existence of some real $\delta(\eta) > 0$ such that, for any measurable subset E of $\mathbb{R}_+$,

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \int_E f_\varepsilon(t,x)\,dx \leq \eta, \tag{6.3.20}$$

as soon as $\text{meas}(E) \leq \delta(\eta)$. Moreover, the Egorov theorem and (6.3.18) imply the existence of a measurable subset E_η of $[0,1/\eta]$ such that

$$\text{meas }(E_\eta) \leq \delta(\eta) \qquad \text{and} \qquad \lim_{k \to +\infty} \sup_{[0,1/\eta] \setminus E_\eta} |\varphi_k - \varphi| = 0.$$

Consequently, for all $t \in (0,T)$, $h \in (-t, T-t)$ and $R \in [0,1/\eta]$, we have

$$\begin{aligned} \left| \int_0^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\,\varphi(x)\,dx \right| \leq & \left| \int_0^R [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\,\varphi_k(x)\,dx \right| \\ & + \left| \int_0^R [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\,[\varphi(x) - \varphi_k(x)]\,dx \right| \\ & + \left| \int_R^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\,\varphi(x)\,dx \right|. \end{aligned}$$

Thus, by the definition of $\delta(\eta)$, E_η and φ_k, we deduce from Lemmas 6.3.1 and 6.3.2 that

$$\begin{aligned}\left|\int_0^\infty [f_\varepsilon(t+h,x)-f_\varepsilon(t,x)]\,\varphi(x)\,dx\right| \;\le\; & \left|\int_t^{t+h}\frac{d}{ds}\left(\int_0^R f_\varepsilon(s,x)\,\varphi_k(x)\,dx\right)\,ds\right|\\ & +2\,M\sup_{[0,R]\setminus E_\eta}|\varphi_k-\varphi| + 4\,\|\varphi\|_{L^\infty}\,\eta\\ & +\frac{4\,\|\varphi\|_{L^\infty}\,M}{R}.\end{aligned}$$

Then, Lemma 6.3.4 ensures that

$$\begin{aligned}\sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T)}\left|\int_0^\infty [f_\varepsilon(t+h,x)-f_\varepsilon(t,x)]\,\varphi(x)\,dx\right| \;\le\; & |h|\,C(\varphi_k)+2\,M\sup_{[0,R]\setminus E_\eta}|\varphi_k-\varphi|\\ & +4\,\|\varphi\|_{L^\infty}\,\eta+\frac{4\,\|\varphi\|_{L^\infty}\,M}{R}.\end{aligned}$$

We let $h\to 0$ and obtain, thanks to Lemma 6.3.4, that

$$\begin{aligned}&\limsup_{h\to 0}\sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T)}\left|\int_0^\infty [f_\varepsilon(t+h,x)-f_\varepsilon(t,x)]\,\varphi(x)\,dx\right|\\ &\qquad\qquad \le 2\,M\sup_{[0,R]\setminus E_\eta}|\varphi_k-\varphi|+4\,\|\varphi\|_{L^\infty}\,\eta+\frac{4\,\|\varphi\|_{L^\infty}\,M}{R}.\end{aligned}$$

We now pass to the successive limits $k\to+\infty$, $\eta\to 0$ and $R\to+\infty$ and deduce that (6.3.15) holds. Therefore, the proof of Lemma 6.3.5 is complete. □

We now check that the function f constructed in Lemma 6.3.5 is a weak solution to the OHS equation. We consider $\varphi\in\mathcal{D}(\mathbb{R}_+)$ and define φ_ε by (6.1.9). It is easily checked that f_ε satisfies, for every $t\in(0,\infty)$,

$$\begin{aligned}&\int_0^\infty f_\varepsilon(t,x)\,\varphi_\varepsilon(x)\,dx-\int_0^\infty f_\varepsilon(0,x)\,\varphi_\varepsilon(x)\,dx\\ &\qquad =\int_0^t\int_0^\infty\int_0^{r_\varepsilon(x)} K_\varepsilon(x,y)\,f_\varepsilon(s,x)f_\varepsilon(s,y)\,[\,y\,D_\varepsilon(\varphi_\varepsilon(x))-\varphi_\varepsilon(y)]\,dy\,dx\,ds. \quad (6.3.21)\end{aligned}$$

It remains now to pass to the limit as $\varepsilon\to 0$ in (6.3.21). For that purpose, we need some convergence results for φ_ε and K_ε.

Lemma 6.3.6 *The sequences φ_ε and K_ε defined by (6.1.9) and (6.2.16) satisfy, for every $R>0$, the following properties:*

$$\|\varphi_\varepsilon\|_{L^\infty}\le\|\varphi\|_{L^\infty} \quad \text{and} \quad \varphi_\varepsilon\longrightarrow\varphi \;\text{ strongly in } L^\infty(\mathbb{R}_+), \quad (6.3.22)$$

$$\|D_\varepsilon(\varphi_\varepsilon)\|_{L^\infty}\le\|\varphi\|_{W^{1,\infty}} \quad \text{and} \quad D_\varepsilon(\varphi_\varepsilon)\longrightarrow\partial_x\varphi \;\text{ strongly in } L^\infty(\mathbb{R}_+), \quad (6.3.23)$$

$$\|K_\varepsilon\|_{L^\infty((0,R)^2)}\le\|K\|_{L^\infty((0,R+1)^2)} \quad \text{and} \quad K_\varepsilon\longrightarrow K \;\text{ a.e. on } \mathbb{R}_+^2. \quad (6.3.24)$$

Proof. Let $x \in \mathrm{supp}(\varphi)$. For ε sufficiently small depending only on $\mathrm{supp}(\varphi)$, there is $i \geq 1$ such that $x \in \Lambda_i^\varepsilon$. Then,

$$\begin{aligned} |D_\varepsilon(\varphi_\varepsilon)(x) - \partial_x\varphi(x)| &= \left| \frac{1}{\varepsilon} \int_{\Lambda_i^\varepsilon} \left[\frac{\varphi(z+\varepsilon) - \varphi(z)}{\varepsilon} - \partial_x \varphi(x) \right] dz \right| \\ &= \left| \frac{1}{\varepsilon^2} \int_{\Lambda_i^\varepsilon} \int_z^{z+\varepsilon} [\partial_x\varphi(w) - \partial_x\varphi(x)]\, dw\, dz \right| \\ &\leq 2\varepsilon \|\varphi\|_{W^{2,\infty}}, \end{aligned}$$

whence (6.3.23). Similar calculations lead to (6.3.22). As for (6.3.24), it readily follows from the definition (6.2.16) of K_ε. □

We next recall the classical following lemma (see, e.g. [8, Lemma A.2] for a proof).

Lemma 6.3.7 *Let U be an open set of $\mathbb{R}^m$, $m \geq 1$, and consider two sequences (v_n) in $L^1(U)$ and (w_n) in $L^\infty(U)$. We suppose that there exist v in $L^1(U)$, w in $L^\infty(U)$ and $C > 0$ such that*

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } L^1(U), \\ &\|w_n\|_{L^\infty} \leq C \quad \text{and} \quad w_n \to w \quad \text{a.e. in } U. \end{aligned}$$

Then

$$\lim_{n\to+\infty} \|v_n(w_n - w)\|_{L^1} = 0 \qquad \text{and} \qquad v_n\, w_n \rightharpoonup v\, w \quad \text{in} \quad L^1(U).$$

We are now in a position to pass to the limit in (6.3.21). Let $\varphi \in \mathcal{D}(\mathbb{R}_+)$ with $\mathrm{supp}(\varphi) \subset [0, L-2]$, for some $L > 2$, and define φ_ε by (6.1.9). Let $T > 0$ and $R > L$. On the one hand, it follows from Lemma 6.3.5 by classical arguments that

$$f_\varepsilon(t,x)\, f_\varepsilon(t,y) \longrightarrow f(t,x)\, f(t,y) \quad \text{in } C([0,T]; w - L^1((0,R)^2)).$$

On the other hand, the definition (6.1.11) of r_ε ensures that $\mathbf{1}_{[0,r_\varepsilon(x)]} \longrightarrow \mathbf{1}_{[0,x]}$ for a.e. $x \in \mathbb{R}_+$, which together with Lemma 6.3.6, implies that

$$\begin{aligned} &K_\varepsilon(x,y)\, [\, y\, D_\varepsilon(\varphi_\varepsilon)(x) - \varphi_\varepsilon(y)]\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y) \\ &\qquad \longrightarrow \quad K(x,y)\, [y\, \partial_x\varphi(x) - \varphi(y)]\, \mathbf{1}_{[0,x]}(y) \qquad a.e. \text{ in } (0,R)^2. \end{aligned}$$

Owing to the bounds on φ_ε, $D_\varepsilon(\varphi_\varepsilon)$ and K_ε in Lemma 6.3.6, we may apply Lemma 6.3.7 to obtain that

$$\begin{aligned} &\int_0^T \int_0^R \int_0^R K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, [\, y\, D_\varepsilon(\varphi_\varepsilon)(x) - \varphi_\varepsilon(y)]\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx\, dt \\ &\qquad \xrightarrow{\varepsilon\to 0} \int_0^T \int_0^R \int_0^R K(x,y)\, f(t,x)\, f(t,y)\, [y\, \partial_x\varphi(x) - \varphi(y)]\, \mathbf{1}_{[0,x]}(y)\, dy\, dx\, dt. \end{aligned}$$

Also, since $\mathrm{supp}(\varphi) \subset [0, R-2]$,

$$\begin{aligned} \iint_{\mathbb{R}_+^2 \setminus [0,R]^2} K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, y\, D_\varepsilon(\varphi_\varepsilon)(x)\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx &= 0, \\ \iint_{\mathbb{R}_+^2 \setminus [0,R]^2} K(x,y)\, f(t,x)\, f(t,y)\, y\, \partial_x \varphi(x)\, \mathbf{1}_{[0,x]}(y)\, dy\, dx &= 0. \end{aligned}$$

Finally, it follows from (6.2.4) and (6.3.13) that

$$\begin{aligned} &\left| \iint_{\mathbb{R}_+^2 \setminus [0,R]^2} K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, \varphi_\varepsilon(y)\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx \right| \\ &\qquad \leq \left| \int_R^\infty dx \int_0^L K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, \varphi_\varepsilon(y)\, dy \right| \\ &\qquad\qquad \leq C\, M^2\, \|\varphi\|_{L^\infty} \sup_{x \geq R} \overline{\omega}_L(x), \end{aligned}$$

and

$$\begin{aligned} &\left| \iint_{\mathbb{R}_+^2 \setminus [0,R]^2} K(x,y)\, f(t,x)\, f(t,y)\, \varphi(y)\, \mathbf{1}_{[0,x]}(y)\, dy\, dx \right| \\ &\qquad \leq \left| \int_R^\infty dx \int_0^L K(x,y)\, f(t,x)\, f(t,y)\, \varphi(y)\, dy \right| \\ &\qquad\qquad \leq C\ M^2\, \|\varphi\|_{L^\infty} \sup_{x \geq R} \omega_L(x). \end{aligned}$$

Therefore, they both tend to 0 as $R \to +\infty$, uniformly with respect to ε.

It remains now to let $\varepsilon \to 0$ in the first two terms of (6.3.21). It readily follows from Lemmas 6.3.2, 6.3.5 and 6.3.6 that

$$\int_0^\infty f_\varepsilon(t,x)\, \varphi_\varepsilon(x)\, dx \longrightarrow \int_0^\infty f(t,x)\, \varphi(x)\, dx,$$

for every $t > 0$. Moreover,

$$f_\varepsilon(0) \longrightarrow f^{in} \qquad \text{in} \qquad L^1(\mathbb{R}_+), \tag{6.3.25}$$

whence, by Lemma 6.3.6,

$$\int_0^\infty f_\varepsilon(0,x)\, \varphi_\varepsilon(x)\, dx \longrightarrow \int_0^\infty f^{in}(x)\, \varphi(x)\, dx. \tag{6.3.26}$$

We thereby obtain that f satisfies (6.2.6) and is, consequently, a weak solution to the OHS equation.

To justify (6.3.25), we first observe that, for $f^{in} \in W^{1,1}(\mathbb{R}_+)$, we have

$$\|f_\varepsilon(0,.) - f^{in}\|_{L^1} \leq \varepsilon\, \|f^{in}\|_{W^{1,1}},$$

whence (6.3.25), for $f^{in} \in W^{1,1}(\mathbb{R}_+)$. The general case for $f^{in} \in L^1(\mathbb{R}_+)$ then follows by a density argument, since

$$\|f_\varepsilon(0,.) - g_\varepsilon(0,.)\|_{L^1} \leq \|f^{in} - g^{in}\|_{L^1},$$

for every $f^{in}, g^{in} \in L^1(\mathbb{R}_+)$.

6.4 The dOHS equation

6.4.1 Proof of Proposition 6.2.4

We are here concerned with the Cauchy problem (6.1.4)-(6.1.6) where the discrete coefficients $K_{i,j}$ satisfy (6.2.17) and the initial data $c^{in} = (c_i^{in})_{i\geq 1}$ satisfies (6.2.18). We proceed as in [3, 14]: we first approximate the dOHS equation by a system of ordinary differential equations.

Let $N \geq 3$ be a positive integer. We consider the following system of N ordinary differential equations:

$$\frac{dc_i^N}{dt} = Q_i^N(c^N), \qquad \text{in } (0,+\infty), \tag{6.4.1}$$

$$c_i^N(0) = c_i^{in}, \tag{6.4.2}$$

for $i \in \{1,\ldots,N\}$, where $c^N = (c_i^N)_{1\leq i\leq N}$ and

$$Q_i^N(c^N) = c_{i-1}^N \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j^N - c_i^N \sum_{j=1}^{i} j\, K_{i,j}\, c_j^N - \sum_{j=i}^{N} K_{i,j}\, c_i^N\, c_j^N. \tag{6.4.3}$$

We first prove the well-posedness of (6.4.1), (6.4.2).

Lemma 6.4.1 *For each $N \geq 3$, there exists a unique non-negative solution $c^N = (c_i^N)_{1\leq i\leq N}$ in $\mathcal{C}^1([0,+\infty);\mathbb{R}^N)$ to the system (6.4.1)-(6.4.3). Moreover, we have*

$$\sum_{i=1}^{N} i\, c_i^N(t) \leq \sum_{i=1}^{N} i\, c_i^{in}, \qquad t \in [0,+\infty). \tag{6.4.4}$$

Proof. Consider $c^{in,N} = (c_i^{in,N}) \in \mathbb{R}^N$. Since Q^N is a locally Lipschitz continuous function, the Cauchy-Lipschitz theorem ensures the existence of a unique maximal solution $c^N = (c_i^N)_{1\leq i\leq N} \in \mathcal{C}^1([0,t^+(c^{in,N}));\mathbb{R}^N)$ to (6.4.1)-(6.4.3), where either $t^+(c^{in,N}) = +\infty$, or $t^+(c^{in,N}) < +\infty$ and

$$\lim_{t\to t^+(c^{in,N})} \sum_{i=1}^{N} |c_i^N(t)| = +\infty.$$

Now, let $c^{in,N} \in [0,+\infty)^N$. Then $c^{in,N} + t\,Q^N(c^{in,N}) \in [0,+\infty)^N$ if t satisfies

$$t\left((N+1) \times \left(\sup_{1\leq i,j\leq N} K_{i,j}\right) \times \sum_{j=1}^{N} c_j^{in,N}\right) \leq 1.$$

Consequently, $\mathrm{dist}(c^{in,N} + t\,Q^N(c^{in,N}), [0,+\infty)^N) = 0$ for t small enough and thus,

$$\liminf_{t\to 0^+} \quad t^{-1}\,\mathrm{dist}\left(c^{in,N} + t\,Q^N(c^{in,N}), [0,+\infty)^N\right) = 0,$$

which corresponds to the *subtangent condition*. Therefore, [2, Theorem 16.5] ensures that, for each $c^{in,N} \in [0,+\infty)^N$, the corresponding maximal solution $c^N = (c_i^N)_{1\leq i\leq N}$ to (6.4.1)-(6.4.3) is non-negative on $[0, t^+(c^{in,N}))$.

Besides, we note that, for every $c \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} i\,Q_i^N(c) = -(N+1)c_N \sum_{j=1}^{N} K_{N,j}\,jc_j.$$

Consequently, we have, for each $t \in [0, t^+(c^{in,N}))$,

$$0 \leq \sum_{i=1}^{N} |c_i^N(t)| = \sum_{i=1}^{N} c_i^N(t) \leq \sum_{i=1}^{N} i\,c_i^N(t) \leq \sum_{i=1}^{N} i\,c_i^{in,N} < +\infty,$$

where $c^{in,N} \in [0,+\infty)^N$ and c^N denotes the corresponding maximal solution to (6.4.1)-(6.4.3). This implies that $t^+(c^{in,N}) = +\infty$ for each $c^{in,N} \in [0,+\infty)^N$ and completes the proof of Lemma 6.4.1. □

It remains now to pass to the limit in (6.4.1)-(6.4.3). To this end, we need some compactness property. By (6.4.4), we already know that $(c_i^N)_{N\geq i}$ is bounded for each $i \geq 1$. We next prove the time equicontinuity of $(c_i^N)_{N\geq i}$.

Lemma 6.4.2 *Let $i \geq 1$. There exists a constant γ_i, depending only on $\sum_{k=1}^{\infty} k\,c_k^{in}$ and i such that, for each $N \geq i$,*

$$\left|\frac{dc_i^N}{dt}\right| \leq \gamma_i, \qquad t \in [0,+\infty). \tag{6.4.5}$$

Proof. Due to (6.2.17), we set, for each $i \geq 1$,

$$\kappa_i := \sup_j \frac{K_{i,j}}{j} < +\infty.$$

Then equations (6.4.1), (6.4.3) and (6.4.4) imply that

$$\begin{aligned}\left|\frac{dc_i^N}{dt}\right| &\leq \kappa_{i-1}\,(i-1)\,c_{i-1}^N \sum_{j=1}^{N} j\,c_j^N + \kappa_i\,i\,c_i^N \sum_{j=1}^{N} j\,c_j^N + \kappa_i\,c_i^N \sum_{j=1}^{N} j\,c_j^N \\ &\leq (\kappa_{i-1} + 2\kappa_i)\left(\sum_{j=1}^{\infty} jc_j^{in}\right)^2.\end{aligned}$$

□

Gathering Lemmas 6.4.1 and 6.4.2, we deduce from the Arzela-Ascoli theorem that there exist a function $c=(c_i)_{i\geq 1}$ and a subsequence of $(c_i^N)_{N\geq i}$, not relabelled, such that

$$c_i^N \longrightarrow c_i \quad \text{in} \quad \mathcal{C}([0,T]), \tag{6.4.6}$$

for all $i\geq 1$ and $T>0$. Then, for each $i\geq 1$, c_i is a non-negative function on $[0,\infty)$ and

$$\sum_{i=1}^{\infty} i\, c_i(t) \leq \sum_{i=1}^{\infty} i\, c_i^{in},$$

for every $t\geq 0$. Consequently, we have

$$\sum_{j=i}^{\infty} K_{i,j}\, c_j(t) \leq \left(\sup_{j\geq i} \frac{K_{i,j}}{j}\right) \times \sum_{j=1}^{\infty} j\, c_j^{in} \leq \kappa_i \sum_{j=1}^{\infty} j\, c_j^{in}, \qquad t\in(0,\infty),$$

whence

$$\sum_{j=i}^{\infty} K_{i,j}\, c_j \in L^1(0,t) \quad \text{for every} \quad t\in(0,\infty).$$

Let $i\geq 1$. We now infer from (6.4.6) that, for every $t\geq 0$,

$$\int_0^t c_{i-1}^N(s) \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j^N(s)\, ds \quad \overset{N\to+\infty}{\longrightarrow} \quad \int_0^t c_{i-1}(s) \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j(s)\, ds,$$

$$\int_0^t c_i^N(s) \sum_{j=1}^{i} j\, K_{i,j}\, c_j^N(s)\, ds \quad \overset{N\to+\infty}{\longrightarrow} \quad \int_0^t c_i(s) \sum_{j=1}^{i} j\, K_{i,j}\, c_j(s)\, ds.$$

It remains only to pass to the limit in the last sum of (6.4.3). To this end, we fix $M\geq i$. For $N>M$, we have

$$\left| \int_0^t \left(\sum_{j=i}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s) - \sum_{j=i}^{\infty} K_{i,j}\, c_i(s)\, c_j(s) \right) ds \right|$$

$$\leq \left| \int_0^t \sum_{j=i}^{M-1} K_{i,j} \left[c_i^N(s)\, c_j^N(s) - c_i(s)\, c_j(s) \right] ds \right| \tag{6.4.7}$$

$$+ \left| \int_0^t \sum_{j=M}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s)\, ds \right| + \left| \int_0^t \sum_{j=M}^{\infty} K_{i,j}\, c_i(s)\, c_j(s)\, ds \right|, \tag{6.4.8}$$

for every $t\geq 0$. By (6.4.6), expression (6.4.7) tends to 0 as $N\to+\infty$. As for (6.4.8), we need the growth assumption of (6.2.17):

$$\int_0^t \sum_{j=M}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s)\, ds \leq \sup_{j\geq M} \frac{K_{i,j}}{j} \int_0^t \sum_{j=M}^{N} c_i^N(s)\, j\, c_j^N(s)\, ds \leq t \left(\sum_{j=1}^{\infty} j\, c_j^{in} \right)^2 \sup_{j\geq M} \frac{K_{i,j}}{j},$$

and, similarly,

$$\int_0^t \sum_{j=M}^{\infty} K_{i,j}\, c_i(s)\, c_j(s)\, ds \leq t \left(\sum_{j=1}^{\infty} j\, c_j^{in} \right)^2 \sup_{j \geq M} \frac{K_{i,j}}{j},$$

for every $t \geq 0$. Letting first $N \to +\infty$ and then $M \to +\infty$, we thus obtain that c satisfies (6.1.4)-(6.1.6). The function c is thus a solution to the dOHS equation in the sense of Definition 6.2.3.

6.4.2 Proof of Proposition 6.2.5

We finally show that the dOHS equation propagates perturbations with an infinite speed.

By (6.1.4), the solution c satisfies, for all $i \geq 1$ and $t \in (0, T)$,

$$c_i(t) = c_i^{in} \exp\left(- \int_0^t E_i(s)\, ds \right) + \int_0^t \exp\left(- \int_s^t E_i(\sigma)\, d\sigma \right) c_{i-1}(s)\, F_i(s)\, ds, \qquad (6.4.9)$$

where

$$E_i(s) = \sum_{j=1}^{i} j\, K_{i,j}\, c_j(s) + \sum_{j=i}^{\infty} K_{i,j}\, c_j(s) \qquad \text{and} \qquad F_i(s) = \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j(s),$$

for every $s \in (0, T)$.

Let us assume that, contrary to our claim, $c_r(\tau) = 0$, for some $r \geq k$ and some $\tau \in (0, T)$. By (6.4.9), $c_r(\tau)$ is the sum of two non-negative terms and thus

$$c_r^{in} = 0 \qquad \text{and} \qquad c_{r-1}\, F_r \equiv 0 \quad \text{on } [0, \tau].$$

If $r = 1$, then $k = 1$ and we have $c_1^{in} = 0$, which contradicts the assumption of Proposition 6.2.5. If $r > 1$, we have in particular that $(r-1)K_{r-1,r-1}c_{r-1}^2 \equiv 0$ on $[0, \tau]$, whence $c_{r-1} \equiv 0$ on $[0, \tau]$. Consequently, the assumption $c_r(\tau) = 0$ implies that

$$c_r^{in} = 0 \qquad \text{and} \qquad c_{r-1}(\tau) = 0.$$

By induction, we deduce that $c_i^{in} = 0$ for every $i \leq r$. In particular, this leads to a contradiction for $i = k$.

6.5 Numerical simulations

In this section, we perform numerical experiments in order to illustrate the convergence in Theorem 6.2.2. We consider the particular case where $K \equiv 1$ on $\mathbb{R}_+^2$ and the initial data is given by

$$F_M^{in} = \frac{2}{M}\, \mathbf{1}_{[0,M]} \quad \text{on } \mathbb{R}_+, \qquad (6.5.1)$$

for some $M > 0$. In that case, there is an explicit solution to the OHS equation, which reads

$$F_M(t,x) = \frac{2}{M(1+t)^2}\,\mathbf{1}_{[0,M]}\left(\frac{x}{1+t}\right), \qquad (t,x)\in\mathbb{R}_+^2.$$

The computational domain is chosen to be $[0,10]$ and we set $M=3$. For any $\varepsilon\in(0,1)$, we define the initial data $c^{in,\varepsilon}=(c_i^{in,\varepsilon})_{i\geq 1}$ for the dOHS equation by (6.2.11), where Λ_i^ε is given by (6.1.8). We next consider the system of ordinary differential equations

$$\frac{dc_i^m}{dt} = \varepsilon\left(c_{i-1}^m\sum_{j=1}^{i-1} j\,c_j^m - c_i^m\sum_{j=1}^{i} j\,c_j^m - \sum_{j=i}^{m} c_i^m\,c_j^m\right) \qquad \text{in } (0,+\infty), \quad (6.5.2)$$

$$c_i^m(0) = c_i^{in}, \quad (6.5.3)$$

for $i\in\{1,\ldots,m\}$, where $m=m(\varepsilon)=[10/\varepsilon-1/2]$ corresponds to the number of cells Λ_i^ε included in the interval $[0,10]$. We next use a Matlab ODE solver to obtain a solution to (6.5.2), (6.5.3) on some time interval $[0,t_{max}]$. The approximated solution f_ε is then given by

$$f_\varepsilon(t,x) = \sum_{i=1}^{m} c_i^m(t)\,\chi_i^\varepsilon(x), \qquad (t,x)\in[0,t_{max}]\times[0,10].$$

The plot of the exact solution F_3 and the approximated solutions f_ε, for $\varepsilon=0.05$, $\varepsilon=0.01$ and $\varepsilon=0.005$ is reported in Figure 6.1 at two different times (for $t_{max}=3$), while the time evolution of the L^1 relative error

$$t\longmapsto \frac{\|F_3-f_\varepsilon\|_{L^1}}{\|F_3\|_{L^1}}(t),$$

is plotted in Figure 6.2. Both figures illustrate the L^1 convergence of f_ε to F_3 as $\varepsilon\to 0$. We point out that the error is concentrated in the neighbourhood of the discontinuity of F_3 (see Figure 6.1 (a)), which was expected since the upwind difference scheme is diffusive and diffusion smears out the discontinuities. A further comment in that direction is that, in Figure 6.2, the L^1 relative error decreases for $t\in[7/3,5/2]$, which can be explained by the fact that the discontinuity of F_3 leaves the computational domain at $t=7/3$. Afterwards, we have $F_3(t)\equiv 2/(3(1+t)^2)$ on $[0,10]$ and the remaining error is mainly due to the truncation of the computational domain. Another source of error comes from Proposition 6.2.5, which states that, contrary to F_3, f_ε is not compactly supported. Consequently, our approximation induces some errors outside the support of the exact solution. Finally, from $t=7/3$, the support of the exact solution F_3 is no more included in the computational domain $[0,10]$ and the approximation will be less and less reliable as t increases.

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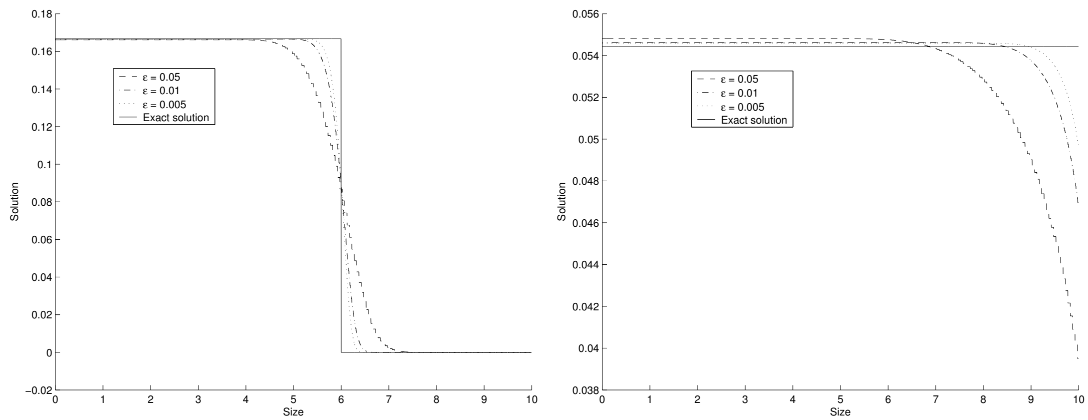

(a) (b)

Figure 6.1: Convergence of solutions to the dOHS equations towards the solution to the OHS equation with initial data (6.5.1) at times $t = 1$ (a) and $t = 2.5$ (b) for $M = 3$

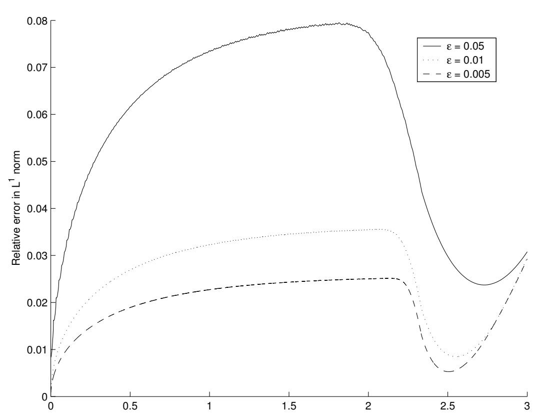

Figure 6.2: Evolution of the relative error in L^1 norm for $M = 3$

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